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Accurate Approximate Regulation of Nonlinear Delay Differential Control Systems

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Abstract— **In this paper we present an approximate controller design methodology for tracking/disturbance-rejection problems governed by nonlinear delay differential control systems. The method considered here is a version of the practical regulation approach developed by the authors in a series of articles. It is important to note that this approach to regulation does not require the existence of an exo-system to define disturbances and signals to be tracked. Therefore, this control law enables tracking and disturbance rejection for general reference and disturbance signals. The idea is similar to the inclusion of a cascade controller design providing a sequence of increasingly more accurate and better preforming controllers. The underlying principle derives from well known geometric methods which rely on the existence of an attractive invariant manifold in the case when the reference and disturbance signals are outputs of an autonomous, linear, neutrally stable exo-system. However, we are able accomplish high performance tracking without this assumption. References in the literature are provided for the history of the methodology and proofs of the error estimates for general systems. We show, in our included example, that the tracking error can be significantly enhanced using a single extra step in the sequence of controllers. In particular, at each step in the cascade controller the error from the previous step provides the reference signal for the next step. In this way, at each step the errors are reduced geometrically.**

I. Introduction and Background

Control systems with delays have a long history and have been studied by many authors (see, [1], [2], [3], [4], [5], [6], [7], [8], [9]) and continues to be an important area of research.

This work is concerned with problems of approximate asymptotic tracking regulation with measured signals, i.e., the case in which the reference and disturbance signals are known for all time. This is the situation with the classical servomechanism tracking problem in which the reference signals are given [10]. If the disturbances are only known to be outputs of a given exo-system, one can introduce a disturbance observer [11] to obtain an asymptotic proxy of the disturbance. Therefore, there is no loss of generality in assuming that the disturbance is given. We also point out that there exist many physical applications where the disturbance can be measured. For example, for a control system consisting of a compressor on a platform, harmonic vibrations provide the main disturbances and they can be measured. The main requirement for both the reference signals and disturbances is that they have some degree of smoothness. The smoother the signals the more accurately they can be tracked or rejected.

The methodology discussed here does not achieve exact asymptotic tracking. Rather, we provide a sequence of controllers, similar in spirit to cascade controllers, where the error at one level becomes the target to track at the next level. In this way we obtain a sequence of controllers which provide increasingly more accurate tracking results [12]. At each step the error is reduced geometrically and seldom more than one or two iterations are required to achieve a desired level of tracking tolerance.

The controller presented in Section IV below has evolved to its current form through a sequence of extensions and enhancements and has been successfully applied to many types of tracking and disturbance rejection problems for distributed parameter systems, see for example, [13], [14], [15] and the references therein. While this new design methodology originally derived from geometric methods involving approximate solution of the regulator equations, it is no longer limited by the restriction that the reference and disturbance signals are generated by an external exo-system. Therefore, we no longer have to deal with solving the regulator equations. Also, for this reason the concept of internal model principle is not applicable. The brief Section IV contains the description of the controller and its subsequent cascade of iterations that produce increasingly more accurate tracking. In that section the regularization parameter $0 < \beta < 1$ and operators $\mathcal{A}_\beta$ and $\mathcal{L}_0$ are defined without much motivation. However, β and the operators $\mathcal{A}_\beta$ and $\mathcal{L}_0$ were derived systematically in the earlier works and are not repeated here due to limited space. For the development and all subsequent details the reader is referred to the survey article [16], which is freely accessible in the arXiv.org e-print archive. This article contains a detailed explanation of the origins of all the operators and equations presented in Section IV. Therefore, Section IV only contains a brief presentation of the *Regularized Dynamic Controller*. We also comment that the controller presented here is not robust with regard to parameters of the system. But the authors are currently developing a robust version of this methodology that will appear in a separate article.

The organization of the paper is as follows. In Section II we formulate the class of nonlinear delay systems as they are usually presented in the literature. In Section III we present the distributed parameter formulation of the delay systems and introduce our main assumptions. In Section IV we present the sequence of regularized dynamic con-

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trollers. See [16] for details on the derivation of this set of approximate controllers. In Section V we present the main estimation of the error obtained in the iterative procedure. Once again, complete details can be found in [13], [14], [15] and an outline can be found in [16]. Finally, in Section VI we present a numerical example for a 1D nonlinear delay equation. Notice that the reference and disturbance signals are smooth, non-periodic and not generated by an autonomous exo-system.

II. Problem Formulation

For a measurable function $x(\cdot)$ on $[-r,+\infty)$, $x_t(\cdot)$ denotes the function in $L^2((-r,0),\mathbb{R}^n)$ given by $x_t(s)=x(t+s)$ for s in $[-r,0]$. We denote by $H^1((-r,0),\mathbb{R}^n)$ the standard Sobolev space. For Banach spaces X and Y we use the notation $\mathcal{B}(X,Y)$ to denote the vector space of bounded linear mappings from X to Y.

We consider nonlinear retarded delay differential control systems of the form

$$\begin{aligned}\dot{x}(t) =&A_0x(t)+A_1x(t-r)+F(x(t))\\ &+Bu(t)+B_{\rm d}d(t). \end{aligned} \tag{1}$$

Here, A_0 and $A_1 \in \mathcal{B}(\mathbb{R}^n,\mathbb{R}^n)$, $B\in\mathcal{B}(\mathbb{R}^m,\mathbb{R}^n)$, $B_{\rm d}\in\mathcal{B}(\mathbb{R}^d,\mathbb{R}^n)$. We assume that the control $u(\cdot)$ and the disturbance $d(\cdot)$ are bounded measurable functions. The nonlinear term $F:\mathbb{R}^n\to\mathbb{R}^n$ is assumed to be a smooth function satisfying $F(0)=0$.

A controlled output is defined by

$$y_c(t)=Cx(t), \tag{2}$$

where $C\in\mathcal{B}(\mathbb{R}^n,\mathbb{R}^m)$ and for $t>0$, $x(t)$ is the solution of system (1). To simplify the presentation in this short paper, we have assumed (without loss of generality) that the number of inputs m is the same as the number of outputs.

In the classical approach of solving an asymptotic regulation problem using geometric methods one must solve the so-called regulator equations (see Lemma 1.3.1. in [17]) in the lumped linear case, [18] for linear distributed parameter systems and [19] for nonlinear distributed parameter systems. In general, these equations are difficult to solve or even to obtain approximate numerical solutions. This is particularly true for distributed parameter systems governed by delay and partial differential equations in higher spatial dimensions, and even more so for nonlinear systems.

From a practical point of view, all that is really needed is to be able to obtain a control law that provides sufficiently accurate tracking and not necessarily exact asymptotic tracking. In a series of recent papers [13], [14], [15] an approximation algorithm was developed for obtaining controllers that deliver highly accurate regulation results, i.e., with very small asymptotic tracking error for general reference and disturbance signals. We make use of these results to address the problem for systems defined by the functional differential equation (1) above.

III. Distributed Parameter Formulation

In order to present the results from [13], [14], [15] for nonlinear delay differential control systems, we first formulate the system (1) within the framework of distributed parameter control systems. This formulation allows one to take advantage of distributed parameter control theory and to employ rigorous numerical schemes that have been developed for delay systems (see [20] and [21]).

It is well known (see Theorem 4.1 on page 46 in [6]) that if one supplies initial conditions

$$x(0)=\eta_0\in\mathbb{R}^n, \tag{3}$$
$$x(s)=\varphi(s),\ s\in[-r,0],\ \varphi(\cdot)\in L^2(-r,0),\mathbb{R}^n), \tag{4}$$

then there exists a unique solution $x(t)\in\mathbb{R}^n$ of (1) and this solution depends continuously on the initial data (see [6], [7]). Thus, a natural state space for the delay differential equation (1) is the product space

$$Z=\mathbb{R}^n\times L^2((-r,0),\mathbb{R}^n) \tag{5}$$

(see [9], [22]). For the linear problem, the system operator $\mathcal{A}$ is defined on the domain

$$\mathcal{D}(\mathcal{A})=\{[\eta\ \ \varphi(\cdot)]^{\intercal}\in Z:\ \varphi(\cdot)\in H^1((-r,0),\mathbb{R}^n),\varphi(0)=\eta\} \tag{6}$$

by

$$\mathcal{A}(z)=[A_0\eta+A_1\varphi(-r)\quad \varphi'(\cdot)]^{\intercal}. \tag{7}$$

In [22] it is shown that $\mathcal{A}$ generates a C_0-semigroup $S(t):Z\to Z$, $t>0$ and

$$S(t)([\eta\quad\varphi(\cdot)]^{\intercal})=[x(t)\quad x_t(\cdot)]^{\intercal}\in Z, \tag{8}$$

where $x(t)$ is the solution to the linearized delay differential equation with initial condition (3)–(4) and $x_t:[-r,0]\to\mathbb{R}^n$ is the "past history" function defined above. Moreover, $S(t)$ is a compact semi-group for all $t\geq r$. If $[\eta\ \ \varphi(\cdot)]^{\intercal}\in\mathcal{D}(\mathcal{A})$, then solutions are piecewise $C^p(0,T)$ for all $T>0$.

The corresponding nonlinear distributed parameter system is defined on Z by

$$\dot{z}(t)=\mathcal{A}z(t)+\mathcal{F}(z(t))+\mathcal{B}u(t)+\mathcal{D}(t), \tag{9}$$

with controlled output

$$y_c(t)=\mathcal{C}z(t). \tag{10}$$

Here, $\mathcal{A}$ is defined by (6)–(7), $\mathcal{B}=[B\quad 0]^{\intercal}$, $\mathcal{D}(t)=[B_{\rm d}d(t)\quad 0]^{\intercal}$, $\mathcal{C}z(t)=Cx(t)$ and $\mathcal{F}:\mathbb{R}^n\times H^1(-r,0)\to\mathbb{R}^n$ is defined by $\mathcal{F}[\eta\quad\varphi(\cdot)]^{\intercal}=[F(\eta)\quad 0]^{\intercal}$.

If $[\eta\quad\varphi(\cdot)]^{\intercal}\in\mathcal{D}(\mathcal{A})$, then solutions $x(t)$ belong to $PWC^p(0,T)$ for all $T>0$ and $p=0,1,2...$, where, for any interval $I\subset(0,\infty)$, $PWC^p(I)$ denotes the space of piecewise C^p functions on I. This means that except for a finite number of points $\{\hat{t}=mr:m\in\mathbb{Z}^+\}\subset I$, $x(t)$ has bounded p^{th} order derivatives. We define a semi-norm on $PWC^p(I)$ by

$$\|\varphi\|_{I,p}=\max_{0\leq j\leq p}\left(\sup_{t\in I,\ t\neq mr}|\varphi^{(j)}(t)|\right). \tag{11}$$

In the case $I=[0,\infty)$ we write $\|\varphi\|_p$. Using this framework we make the following assumptions:

A1 The signals $y_{ref}(\cdot)$ and $d(\cdot)$ are smooth functions of time on $(0,+\infty)$.

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A2 The state operator $\mathcal{A}$ is the generator of an exponentially stable semigroup on the Hilbert space $Z = \mathbb{R}^n \times L^2(-r,0)$.

A3 *The classical non-resonance condition:* Let $\mathcal{G}(s) = \mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}$ denote the transfer function of the linear plant. We assume

$$\mathcal{G} := \mathcal{G}(0) = \mathcal{C}(-\mathcal{A}^{-1})\mathcal{B}$$

is invertible.

Remark 1: Observe that assumption **A1** is very weak and is always satisfied if the reference and disturbance signals are defined by a finite dimensional linear exogenous system. Also, one can replace **A2** by the assumption that the system is stabilizable. In this case one could first find a stabilizing state feedback $\mathcal{K}$ and then replace $\mathcal{A}$ by $\mathcal{A}+\mathcal{B}\mathcal{K}$. While the stabilization problem is certainly important, it is not the main concern in this work. In any case, conditions **A1** - **A3** are not very restrictive and are typically satisfied for the tracking problems considered here.

IV. Regularized Dynamic Controller

Due to the special form of the delay system (1)–(2) and its distributed parameter formulation (9), it is possible to introduce a somewhat simplified version of the method described in [13] and [15]. As has already been mentioned in the introduction, since the publication of these earlier results, the authors have simplified the formulation and presentation of the main parts of their work. Namely, the notions of *Dynamic Regulator Equations* and *Regularized Dynamic Regulator Equations* have been replaced by the more elegant concept of *Regularized Dynamic Controller*. While the new formulation is nicer and simpler, it is completely equivalent to the one given in the earlier works. As mentioned in the introduction, in order for the reader to be able to understand the evolution of the ideas leading to the *Regularized Dynamic Controller* the authors have published a summary of the information on arXiv [16].

Recall that the main objective in this work is to present the approximate controller and then demonstrate its functionality in a numerical example.

In the following, a parameter β is introduced to provide a numerically stable replacement for a certain singular system. Smaller values produce more accurate tracking but there is a limit to how small it can be chosen without producing an unstable system. Again, the details may be found in [16].

With this understanding, for a fixed regularization parameter $0<\beta<1$, define

$$\mathcal{A}_\beta = \left(\mathcal{A} - \zeta\mathcal{B}\mathcal{G}^{-1}\mathcal{C}\right), \quad \zeta = \frac{(1-\beta)}{\beta}. \tag{12}$$

Note that $\zeta\mathcal{B}\mathcal{G}^{-1}\mathcal{C}$ is a compact perturbation of $\mathcal{A}$ and hence for β sufficiently close to 1, the operator $\mathcal{A}_\beta$ generates an exponentially stable semigroup $S_\beta(t)$. Also, define the operator $\mathcal{L}_0 : Z \to Z$

$$\mathcal{L}_0 = (I + \mathcal{B}\mathcal{G}^{-1}\mathcal{C}\mathcal{A}^{-1}). \tag{13}$$

The *Regularized Dynamic Controller*, as described in [16], consists of a sequence of approximate controllers, reminiscent of a cascade controller, where at each step the error obtained in a given step becomes the reference signal to be tracked at the next step. In the zeroth step, we choose an arbitrary small initial condition z_0 and solve the *Regularized System* (14) – (15) given below for the state variable $\bar{z}^0$. The resulting control, $\bar{u}^0$, is then obtained as an output of the system in (16).

$$\dot{\bar{z}}^0(t) = \mathcal{A}_\beta \bar{z}^0(t) + \mathcal{L}_0\left(\mathcal{F}(\bar{z}^0) + \mathcal{D}(t)\right) + \frac{1}{\beta}\mathcal{B}y_{ref}(t), \tag{14}$$

$$\bar{z}^0(0) = z_0 \in Z, \tag{15}$$

$$\bar{u}^0(t) = \mathcal{G}^{-1}\left[\frac{1}{\beta}y_{ref}(t) - \frac{(1-\beta)}{\beta}\mathcal{C}\bar{z}^0(t) + \mathcal{C}\mathcal{A}^{-1}\left(\mathcal{F}(\bar{z}^0(t)) + \mathcal{D}(t)\right)\right]. \tag{16}$$

Next, for fixed $n>0$, we define the series of state variables and controls,

$$\bar{z}_n = \sum_{j=0}^{n} \bar{z}^j, \quad \bar{u}_n = \sum_{j=0}^{n} \bar{u}^j, \tag{17}$$

where, for all $0<j\le n$, the state variable and control, $\bar{z}^j$ and $\bar{u}^j$, satisfy the *Regularized System*

$$\dot{\bar{z}}^j(t) = \mathcal{A}_\beta \bar{z}^j(t) + L_0\mathbb{F}_j + \frac{1}{\beta}\mathcal{B}e_{j-1}(t), \tag{18}$$

$$\bar{z}^j(0) = 0, \tag{19}$$

$$\bar{u}^j(t) = \left[\frac{1}{\beta}e_{j-1}(t) - \frac{(1-\beta)}{\beta}\mathcal{C}\bar{z}^j(t) + \mathcal{C}\mathcal{A}^{-1}\mathbb{F}_j\right]. \tag{20}$$

Here the nonlinear terms are given by

$$\mathbb{F}_j = \mathcal{F}(\bar{z}_j(t)) - \mathcal{F}(\bar{z}_{(j-1)}(t)), \tag{21}$$

and the sequence of errors $e_j(t)$ are defined by

$$e_0 = y_{ref}(t) - \mathcal{C}\bar{z}^0, \tag{22}$$

$$e_j(t) = e_{j-1}(t) - \mathcal{C}\bar{z}^j, \text{ for } j \ge 1. \tag{23}$$

Solving the cascade of systems for $j = 0,\cdots,n$ we obtain the desired approximate control $\bar{u}_n$, as described in (17), which delivers the error $e_n(t)$ in (23), for $j=n$.

V. Estimate of the Error

In order to briefly describe the error analysis found in [13], [14], [15], [16] let us introduce the following operators

$$K(t) = -\frac{1}{\beta}\mathcal{C}\mathcal{A}_\beta^{-1}e^{\mathcal{A}_\beta t}\mathcal{B}, \tag{24}$$

$$K_d(t) = -\mathcal{C}\mathcal{A}_\beta^{-1}e^{\mathcal{A}_\beta t}\mathcal{L}_0\mathcal{D}, \tag{25}$$

$$H(t) = -\mathcal{C}e^{\mathcal{A}_\beta t}\mathcal{L}_0, \quad \mathcal{L}_0 = (I + \mathcal{B}\mathcal{C}\mathcal{A}^{-1}). \tag{26}$$

To simplify many of the estimates obtained later we define the following class of functions.

Definition 1: We denote by $\mathcal{H}$ all functions $h(t)$ in the form $p(t)e^{-\omega_\beta t}$ with $\omega_\beta > 0$ and $p(t)$ is a polynomial in t. Notice that functions in the set $\mathcal{H}$ go to zero exponentially fast as t goes to infinity.

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Using the operators given in (24) – (26) we can present the error estimates using explicit formulas for the errors. Namely, from the first controller (14) – (16) we obtain.

$$e_0(t) = H_0(t) + (K * y'_{ref})(t) + (K_d * d')(t) + (H * \mathcal{F}_0)(t), \quad (27)$$

where we have introduced the convolution operator defined for any two functions f and g by

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau,$$

and where

$$H_0(t) := \mathcal{C} e^{\mathcal{C}_\beta t} \bar{z}_0 - \mathcal{C} \mathcal{A}_\beta^{-1} e^{\mathcal{A}_\beta t} \left[\frac{1}{\beta} \mathcal{B} y_{ref}(0) + \mathcal{L}_0 \mathcal{D}(0) \right] \quad (28)$$

decays to 0 exponentially as t goes to infinity and is it clearly bounded above by some function $h_0 \in \mathcal{H}$.

Similarly, for any $j \geq 1$, the controllers (18) – (20) produce the following explicit formulas relating $e_j(t)$ to $e_{j-1}(t)$.

$$e_j(t) = H_j(t) + (K * e'_{j-1})(t) + (H * \mathbb{F}_j)(t), \quad (29)$$

and again

$$H_j(t) := -\mathcal{C} \mathcal{A}_\beta^{-1} e^{\mathcal{A}_\beta t} \left[\frac{1}{\beta} \mathcal{B} e_{j-1}(0) \right] \quad (30)$$

is bounded above by some function $h_j \in \mathcal{H}$.

Using a modification of the arguments given in Section 5.2 in [14], we obtain the analog of Theorem 4 in [14] (see also [16]). In particular, we had to modify the proof of Theorem 4 to the case where (11) only defines a semi-norm on $PWC^p(0,T)$ to allow for "jumps" in the derivatives of the solution $x(t)$ that might occur at multiples of the time delay r.

Theorem 1: Let ε denote the Lipschitz constant for $\mathcal{F}$ in Z. If

$$D = \int_0^\infty \|K(t)\|\, \mathrm{d}t, \qquad D_d = \int_0^\infty \|K_d(t)\|\, \mathrm{d}t, \quad (31)$$

$$\widetilde{D} = \int_0^\infty \|\mathcal{C} e^{\mathcal{A}_\beta t} \mathcal{L}_0\|\, \mathrm{d}t, \quad (32)$$

$$\mathcal{K} = \int_0^t \|e^{\mathcal{A}_\beta t} \mathcal{B}\|\, \mathrm{d}t \ , \ \mathcal{P} = \int_0^t \|e^{\mathcal{A}_\beta t} \mathcal{L}_0\|\, \mathrm{d}t \ , \quad (33)$$

$$\mathcal{D} = \widetilde{D} \left(\frac{\mathcal{K}\varepsilon}{1 - \varepsilon \mathcal{P}} \right), \quad \mathcal{D}_d = \widetilde{D}_d \left(\frac{\mathcal{P}\varepsilon}{1 - \varepsilon \mathcal{P}} \right), \quad (34)$$

then

$$\limsup \|e_0\| \leq (D + \mathcal{D}) \limsup \|y_{ref}\|_1 + (D_d + \mathcal{D}_d) \limsup \|d\|_1, \quad (35)$$

$$\limsup \|e_1\| \leq (D + \mathcal{D})^2 \limsup \|y_{ref}\|_2 + (D_d + \mathcal{D}_d)^2 \limsup \|d\|_2. \quad (36)$$

For $n > 1$ the calculations become much more involved and we do not have simple estimates like the ones in (35) and (36). For example, at the third iteration, i.e., for $n = 2$, we do obtain

$$\limsup \|e_2\| \leq (D + \mathcal{D})^3 \limsup \|y_{ref}\|_3 + (D_d + \mathcal{D}_d)^3 \limsup \|d\|_3 + V_2,$$

where V_2 is a small (but complicated) term.

We note that, in the linear case, i.e., when $F = 0$ in (1), Theorem 6.1 of Section 6 in [16] establishes that the sequence of controls $\overline{u}_N$ defined in (17) do indeed converge to the control u that would produce exact asymptotic tracking.

VI. Numerical Example

Example 6.1 (1D Delay System): In this example we consider a problem of tracking regulation with disturbance rejection for a 1D nonlinear delay differential equation. A detailed discussion of stability for this model can be found in [6], [7], [8]. For this example, the reference and disturbance signals are rather complicated and are not generated by an exo-system as in the case of classical geometric regulator theory. Thus, we demonstrate that this new design strategy allows for tracking and rejection of very general signals that can not be handled by classical methods.

The numerical results are based on the standard averaging / finite volume approximation scheme described in [9], [23].

The state equation is given by

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-r) - x(t)\sin(x(t)) + bu(t) + d(t), \quad (37)$$

with output

$$y_c(t) = cx(t). \quad (38)$$

Here $F(x) = x\sin(x)$ is an analytic function. For this example we set

$$a_0 = -10, \quad a_1 = -1, \quad b = 1, \quad c = 1,$$

$$r = 1, \varphi(s) = 0.5, \ -r < s \leq 0.$$

We consider the problem of tracking a reference signal $y_{ref}(t)$, depicted in Figure 1, and given explicitly by

$$y_{ref}(t) = \sin(\alpha(t)\sqrt{t+1}), \quad (39)$$

where

$$\alpha(t) = \frac{1}{2}\left(3 + \sin\left(\frac{\pi t}{20}\right)\right) \quad (40)$$

is a smooth function, giving a time dependent frequency that oscillates between 1 and 2. Clearly, y_{ref} is a smooth bounded function that is not periodic. Notice that $y_{ref}(0) = \sin(3/2) = 0.9975$, which is not 1 as it appears to be in Figure 1.

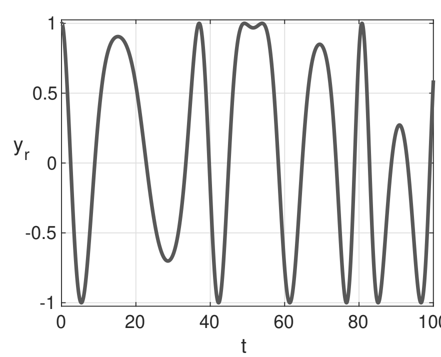

Fig. 1. Reference signal $y_{\mathrm{ref}}(t)$ for $0 < t < 100$.

Similarly, we consider the disturbance $d(t)$ in (37) to be given by

$$d(t) = \mathrm{e}^{\sin(\sqrt{(t+1)})} \sin(t), \quad (41)$$

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and is shown in Figure 2, below. Notice that these functions are not solutions of a simple exo-system as in the classical literature. Observe, also, that assumptions **A1–A3** hold and $\mathcal{G}^{-1} = -[a_0 + a_1]^{-1} = 1/3$.

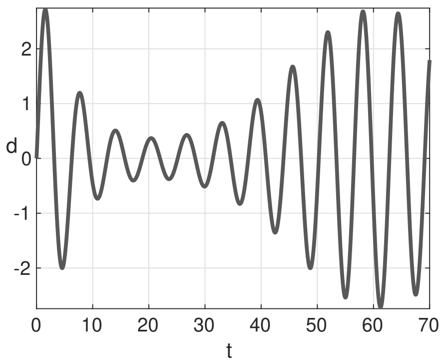

Fig. 2. The disturbance function $d(t)$.

To demonstrate the applicability of the method we ran numerical simulations using three different values of the regularization parameter, $\beta = 0.75, 0.25$ and 0.1. For each β, the control $u(t) = u_\beta(t)$ used in solving the system (37) is $\overline{u}_n(t)$ obtained from applying the cascade controller described in Section IV for $n = 1$.

In each case we started with a constant initial condition, resulting in a distributed parameter system defined by

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{F}(z(t)) + \mathcal{B}u_\beta(t) + \mathcal{D}(t), \tag{42}$$

$$z(0) = \begin{bmatrix} \eta_0 \\ \varphi(\cdot) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}. \tag{43}$$

The main impact of the choice of an initial conditions is that the transient values near $t = 0$ are different; but the asymptotic values will be essentially the same. All errors are reported in Table I for both first step ($e_0(t)$) and the second step ($e_1(t)$), and for the closed loop system ($e(t)$). The results clearly show that after two steps the errors are greatly reduced and that the error obtained in the closed loop system, denoted by $e(t)$, matches almost exactly with the error $e_1(t)$. One can also see the advantage of choosing smaller values of β. However, when β becomes too small, the regularized dynamic controller system becomes unstable. Therefore, there is a trade-off between improved accuracy and stability. The table also compares the theoretical bounds given in (36) with the actual computed error for the full nonlinear problem. An interesting result for this example is that, using our control, the disturbance does not effect the error estimate. In particular, for this example $K_d(t) = 0$ in (25) and so is $\mathcal{K} = 0$ in (33). Thus, the disturbance $d(t)$ has no impact the error bounds, even though it plays an important role in the design of the controller u_β, and it cannot be ignored.

Therefore the estimates in (35) and (36) are much simpler. Namely,

$$\limsup \|e_0\| \le \limsup \|y_r\|_1 := L_1 D,$$
$$\limsup \|e_1\| \le \limsup \|y_r\|_2 := L_2 D^2.$$

For our particular y_{ref} and d we have

$$\max_{j=0,1,2} \sup_{[0,\infty)} |y_{ref}^{(j)}(t)| = 1, \quad \max_{j=0,1,2} \sup_{[0,\infty)} |d^{(j)}(t)| = \mathrm{e}.$$

In Figures 3 and 4, the reference signal $y_{ref}(t)$ is compared to the controlled output $y_c(t)$, for $\beta = 0.25$. In Figure 3, we plotted $y_{ref}(t)$ for $0 < t < 70$ and $y_c(t)$ for $-1 < t < 70$, and the two are almost indistinguishable. In Figure 4, we zoomed around $-1 < t < 2$. The largest difference occurs and is distinguishable only in the proximity of 0. In Figure 5 we plot the control $u_\beta(t)$.

TABLE I

COMPARISON $\limsup|e(t)|$, $\limsup|e_j(t)|$ AND L_1D^2 FOR VARYING β

	$\beta = 0.75$	$\beta = 0.25$	$\beta = 0.1$
$\limsup\|e_0(t)\|$	1.2912e-02	4.2582e-03	1.6979e-03
$\limsup\|e_1(t)\|$	2.1045e-04	2.2689e-05	3.6018e-06
$\limsup\|e(t)\|$	2.1045e-04	2.2689-05	3.6018e-06
L_2D^2	2.0272e-02	2.4313e-03	4.1351e-04

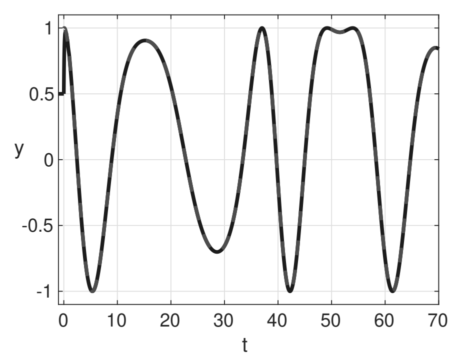

Fig. 3. $y_{ref}(t)$ (dashed) and $y_c(t)$ (solid) for $0 < t < 70$, $\beta = 0.25$.

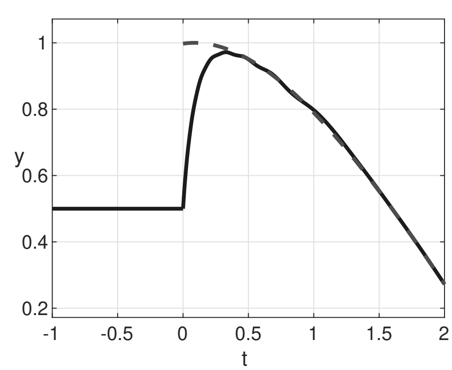

Fig. 4. $y_{ref}(t)$ (dashed) and $y_c(t)$ (solid) for $t \le 2$, $\beta = 0.25$.

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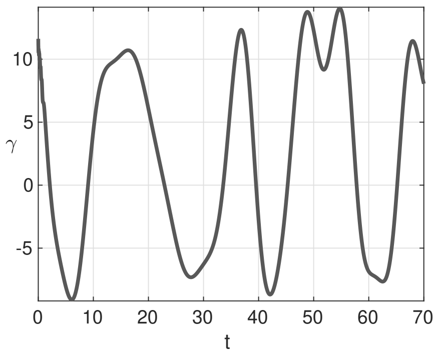

Fig. 5. Plot of the Controller $u_\beta(t)$, $\beta = 0.25$.

Figure 6 demonstrates the effectiveness of applying one extra step in the iterative controller process. In this figure we plotted $e_0(t)$ vs $e_1(t)$, and e_1 is in general orders of magnitude smaller than e_0. In order to better isolate the asymptotic behavior of e_0 and e_1, we plotted only times grater than 5, when all transient contributions to the errors have already exponentially decayed close to zero.

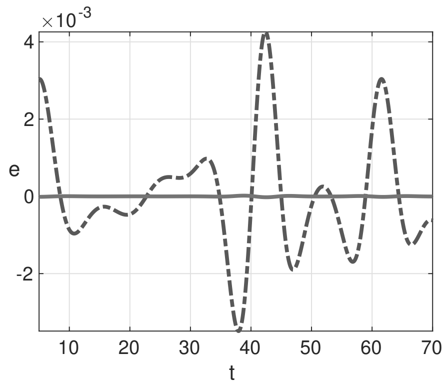

Fig. 6. Plot of $e_0(t)$ (dashed) and $e_1(t)$ (solid) for $5 < t < 70$, $\beta = 0.25$.

VII. Conclusion and Future work

In this paper we used a modified version of a regularization methodology to provide a "cascade" type control law to improve accuracy for a class of nonlinear tracking problems defined by delay differential equations. Although the approach here could be developed without first formulating the problem as an infinite dimensional distributed parameter system, this formulation allows for a more direct application of the existing theory in [13] and [15] and can be easily implemented by employing standard approximation of delay control systems. In addition, it is not necessary for the disturbance or reference signals to be generated by an exogenous system. The corresponding dynamic control law is equivalent to a regularized well-posed delay differential equation which in turn defines the controller. An example was given to illustrate the method and to show typical performance achieved by this method.

For the linear case, in the paper [15] the authors present a complete set of errors estimates for $e_n(t)$ for all n. The problem of obtaining a complete set of rigorous error estimates for the full nonlinear system remains open. However, the numerical results in [13] for PDE systems and the example presented here for delay systems suggest it should be possible to obtain nonlinear error estimates. The authors continue to work on this problem.

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