

# Persistence of Excitation in Uniformly Embedded Reproducing Kernel Hilbert (RKH) Spaces

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**Abstract**—This paper introduces two new notions of persistence of excitation (PE) in reproducing kernel Hilbert spaces (RKHS) that can be used to establish convergence of function estimates generated by the RKHS embedding method. The two PE conditions are proven to be equivalent provided a type of uniform equicontinuity holds for the composition operator  $g \mapsto g \circ x$ , where  $t \mapsto x(t)$  is the unknown state trajectory. The paper then establishes sufficient conditions for the uniform asymptotic stability (UAS) of the error equations of RKHS embedding in term of these PE conditions. The proof is self-contained and treats the general case. Numerical examples are presented that illustrate qualitatively the convergence of the RKHS embedding method where function estimates converge over the positive limit set, a smooth, regularly embedded submanifold of the state space.

## I. INTRODUCTION

Adaptive online estimation for uncertain systems governed by nonlinear ordinary differential equations (ODEs) is now a classical topic in estimation and control theory. Systematic study of this topic has a long history, and many of the first principles can be found in texts on adaptive estimation and control theory [1]–[3]. In general, convergence of state estimates in such schemes is easier to establish than to guarantee parameter convergence. Parameter convergence refers here to estimates of the (real) constants that characterize an unknown function appearing in the uncertain governing ODEs. Beginning with analyses such as in [3]–[5], sufficient conditions for parameter convergence in terms of various definitions of persistence of excitation in finite dimensional state spaces have been studied carefully. These initial investigations have inspired numerous generalizations of PE conditions for evolution laws in  $\mathbb{R}^d$ , with notable examples including [6]–[9]. The analysis of PE conditions is further extended for evolution equations defined in terms of a pivot space structure in Banach and Hilbert spaces in references [10]–[14].

This paper studies novel persistence of excitation (PE) conditions that play a role in recently introduced method of reproducing kernel Hilbert space (RKHS) embedding for adaptive estimation of uncertain, nonlinear ODE systems [15]–[17]. The RKHS embedding method analyzes the uncertain nonlinear system of ODEs by replacing them with a distributed parameter system (DPS). While the usual approach such as in the texts above describe evolution of states and parameters in the finite dimensional space  $\mathbb{R}^d \times \mathbb{R}^n$ , the

RKHS embedding method considers evolution of states and function estimates in  $\mathbb{R}^d \times H$  with  $H$  an infinite dimensional RKHS of functions.

Some of the key theoretical questions regarding the RKHS embedding method have been studied in references [15]–[19]. The well-posedness of the infinite dimensional evolution is studied in [15], [16], including the study of the existence and uniqueness of solutions and their continuous dependence on the initial conditions. To implement the RKHS embedding method in practice, the finite dimensional approximation is necessary. Elementary convergence results for approximation are obtained in [15], [16]. In the subsequent discussion [18], the relationship between the PE condition in RKHS and the positive limit set of the unknown system is studied, including some cases when the positive limit set is a manifold. As for practical application, an early example is given in [19] where the strategy of RKHS embedding is applied to an  $\mathcal{L}^1$  adaptive control problem. In reference [17], the basic theory is extended and adapted to construct a consensus estimator, and that analysis extends the method to the estimation of vector-valued functions.

For classical adaptive estimation, the PE condition is understood as ensuring the positivity of integrated regressors over parameter space. In the RKHS embedding method, the PE condition is cast in term of the evaluation functional  $\mathcal{E}_x : f \mapsto f(x)$  for  $x \in X$ . Here we study various alternative statements of the PE conditions in the class of *uniformly embedded* RKHS. When the RKHS is uniformly embedded in the space of continuous functions, we find that many of the well-known classical statements about the PE condition in finite dimensional spaces have analogous counterparts in the infinite dimensional RKHS. The PE condition is established as a sufficient condition for the UAS of the error equations for the RKHS embedding method in [15]–[17], [19], in some instances. However, the conclusion there is proven only for a very specific case, and the authors have found that one of the simpler proofs in [15] is unfortunately incorrect. In this paper, we find an alternate self-contained proof which treats the general case.

### A. Overview of New Results

The conventional notion of persistence of excitation is defined to study the uniform asymptotic stability of error equations that have the form

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{\alpha}}(t) \end{bmatrix} = \begin{bmatrix} A & B\Phi^T(x(t)) \\ -\mu\Phi(x(t))B^T P & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{\alpha}(t) \end{bmatrix}, \quad (1)$$

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with  $\tilde{x}(t) = x(t) - \hat{x}(t) \in \mathbb{R}^d$  the error in state estimates  $\hat{x}(t)$  of the true trajectory  $x(t)$ ,  $\tilde{\alpha}(t) = \alpha^* - \hat{\alpha}(t) \in \mathbb{R}^n$  the error in the parameter estimates  $\hat{\alpha}(t)$  of the true parameters  $\alpha^*$ ,  $A \in \mathbb{R}^{d \times d}$  a Hurwitz matrix,  $B \in \mathbb{R}^{d \times 1}$ ,  $P \in \mathbb{R}^{d \times d}$  the unique positive definite solution of the Lyapunov equation  $A^T P + P A = -Q$  for some fixed positive definite  $Q \in \mathbb{R}^{d \times d}$ , and  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^n$  a collection of regressors for the system. The associated, conventional PE condition follows in Definition 1. This condition is a sufficient condition for the UAS of the error Equations 1. Details are discussed in reference [3]–[9].

*Definition 1: (PE in  $\mathbb{R}^n$ )* A trajectory  $t \mapsto x(t) \in \mathbb{R}^d$  persistently excites a family of regressor functions  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^n$  if there exist constants  $T_0, \Delta, \gamma > 0$  such that

$$\int_t^{t+\Delta} v^T \Phi(x(\tau)) \Phi^T(x(\tau)) v d\tau \geq \gamma > 0 \quad (2)$$

for each  $t \geq T_0$  and  $v \in \mathbb{R}^n$  with  $\|v\| = 1$ .

In this paper, we introduce two new definitions of persistence of excitation that are used in conjunction with the RKHS embedding method.

*Definition 2: (PE.1)* The trajectory  $t \mapsto x(t) \in \mathbb{R}^d$  persistently excites the indexing set  $\Omega$  and the RKHS  $H_\Omega$  provided there exist positive constants  $T_0, \gamma, \delta$ , and  $\Delta$ , such that for each  $t \geq T_0$  and any  $g \in H_\Omega$  with  $\|g\|_{H_\Omega} = 1$ , there exists  $s \in [t, t + \Delta]$  such that

$$\left| \int_s^{s+\delta} \mathcal{E}_{x(\tau)} g d\tau \right| \geq \gamma > 0. \quad (3)$$

*Definition 3: (PE.2)* The trajectory  $t \mapsto x(t) \in \mathbb{R}^d$  persistently excites the indexing set  $\Omega$  and the RKHS  $H_\Omega$  provided there exist positive constants  $T_0, \gamma$ , and  $\Delta$ , such that

$$\int_t^{t+\Delta} \left( \mathcal{E}_{x(\tau)}^* \mathcal{E}_{x(\tau)} g, g \right)_{H_\Omega} d\tau \geq \gamma > 0 \quad (4)$$

for all  $t \geq T_0$  and any  $g \in H_\Omega$  with  $\|g\|_{H_\Omega} = 1$ .

These two definitions are analogous to those studied in the classical scenario in [3], [4], but here they are expressed in terms of the evaluation operator  $\mathcal{E}_x$  rather than the regressor functions  $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^n$ . We define the composition operator  $\mathcal{U}$  associated with the trajectory  $t \mapsto x(t)$  to be the mapping  $\mathcal{U} : g \mapsto g \circ x$ . Let  $\bar{S}_1 := \{g \in H_\Omega : \|g\|_{H_\Omega} = 1\}$  denote the unit sphere in RKH subspace  $H_\Omega$ . The first primary result of this paper is stated in terms of the collection of functions  $\mathcal{U}(\bar{S}_1) := \{g(x(\cdot)) : \|g\|_{H_\Omega} = 1, g \in H_\Omega\}$ . We establish in Theorems 1 and 2 that

$$\text{“PE.1} \implies \text{PE.2,”} \quad \text{and}$$

$$\text{“PE.2 and } \mathcal{U}(\bar{S}_1) \text{ uniformly equicontinuous} \implies \text{PE.1.”}$$

This theorem can be viewed as a type of generalization of the results in [3], [4] to the DPS that arises from the RKHS embedding method. Essentially, the assumption that  $\mathcal{U}(\bar{S}_1)$  is uniformly equicontinuous eliminates in the infinite dimensional case the possibility of “pathological” rapid switching that has been studied and commented on in detail in [3], [4] for the finite dimensional case.

The role of these PE conditions above is studied for the following DPS that is associated with estimation errors in the RKHS embedding formulation:

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{f}}(t) \end{bmatrix} = \begin{bmatrix} A & B\mathcal{E}_{x(t)} \\ -\mu(B\mathcal{E}_{x(t)})^*P & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{f}(t) \end{bmatrix}. \quad (5)$$

Here  $\tilde{x}(t) \in \mathbb{R}^d$ ,  $A \in \mathbb{R}^{d \times d}$  Hurwitz, and  $B \in \mathbb{R}^{d \times 1}$  are defined as above, but now  $\tilde{f}(t) = f - \hat{f}(t) \in H_\Omega$  is the error of the function estimates  $\hat{f}(t)$  of the true function  $f$ . The second fundamental result of this paper is a detailed proof in Theorem 3 of the fact that

$$\text{“PE.1} \implies \text{the error equations are UAS in } \mathbb{R}^d \times H_\Omega\text{”}.$$

We should note that references [15]–[17] establish this fact as a special case of the much more general analysis in [10] when  $P = I$  and  $A$  is in fact negative definite, but here we treat the general situation. Also, we feel that the proofs in this paper are substantially simpler than that in [10], more closely resemble the classical analyses in [3], [4], and are of independent interest.

## B. Notation

In this paper,  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^d$ ,  $\|\cdot\|_{op}$  is the operator norm, and  $\|\cdot\|_{H_X}$  is the norm on a Hilbert space  $H_X$  of real-valued functions over  $X$ . The inner product on  $\mathbb{R}^d$  and  $H_X$  are written as  $(\cdot, \cdot)$  and  $(\cdot, \cdot)_{H_X}$  respectively. We define  $\|\cdot\|_{H_\Omega} := \|P_\Omega(\cdot)\|_{H_X}$  where  $P_\Omega$  is the  $H_X$ -orthogonal projection onto  $H_\Omega \subseteq H_X$ . Note that  $\|\cdot\|_{H_\Omega}$  is a norm in the RKH subspace  $H_\Omega$ .

## II. RKHS EMBEDDING METHOD

### A. Review of RKHS

To define the RKHS embedding method, we briefly review some of the defining properties of an RKHS. A real RKHS  $H_X$  over a subset  $X$  is a Hilbert space of functions over  $X$ . It is defined in terms of an admissible kernel  $\mathfrak{K}_X : X \times X \rightarrow \mathbb{R}$  that has what is known as the reproducing property. This property is given in terms of the basis function  $\mathfrak{K}_{X,x}$  centered at  $x \in X$ , which is defined from the identity  $\mathfrak{K}_{X,x}(\cdot) := \mathfrak{K}_X(x, \cdot)$ . The kernel  $\mathfrak{K}_X$  is said to have the reproducing property provided  $f(x) = \mathcal{E}_x f = (\mathfrak{K}_{X,x}, f)_{H_X}$  for each  $x \in X$  and  $f \in H_X$ . The RKHS  $H_X$  is then defined as the closed linear space  $H_X = \overline{\text{span}\{\mathfrak{K}_{X,x} : x \in X\}}$  [20].

When  $X$  is a subset of  $\mathbb{R}^d$ , or when  $X$  happens to be a certain type of manifold, many choices of admissible kernels exist. Among all the popular choices, the Gaussian kernel might be the most well-known kernel. See [21] for summaries of possible kernels over (subsets of)  $\mathbb{R}^d$  and [22] for kernels over some choices of manifolds. In this paper, we only consider the RKHS for which we have the uniformly continuous embedding  $H_X \hookrightarrow \mathcal{C}(X)$ . This embedding holds provided that there is a constant  $c > 0$  such that  $\|f\|_{\mathcal{C}(X)} \leq c\|f\|_{H_X}$  for all  $f \in H_X$ . Therefore, a sufficient condition for uniform embedding is that a constant  $\bar{k}$  exists such that

$\mathfrak{K}_X(x, x) \leq \bar{k}^2 < \infty$  for all  $x \in X$ . In this case we have

$$\begin{aligned} |f(x)| &= |\mathcal{E}_x f| = |(\mathfrak{K}_{X,x}, f)_{H_X}| \leq \|\mathfrak{K}_{X,x}\|_{H_X} \|f\|_{H_X} \\ &\leq \sqrt{\mathfrak{K}_X(x, x)} \|f\|_{H_X} \leq \bar{k} \|f\|_{H_X}. \end{aligned}$$

The condition that  $\mathfrak{K}_X(x, x) \leq \bar{k}^2$  thereby guarantees that  $\|\mathcal{E}_x\|_{op} \leq \bar{k}$ , that is, the evaluation operator is uniformly bounded in  $x \in X$ . This property will be frequently used in the following proofs.

When  $H_X$  is an RKHS, we can define the closed subspace  $H_\Omega = \overline{\text{span}\{\mathfrak{K}_{X,x} : x \in \Omega\}}$  when  $\Omega \subseteq X$ . The subset  $\Omega$  is also called the *indexing set* of the RKHS  $H_\Omega \subseteq H_X$ . We define the orthogonal decomposition  $H_X = H_\Omega \oplus V_\Omega$  with  $V_\Omega = H_\Omega^\perp$ . Using the definition of RKHS  $H_\Omega$  and the reproducing property, it is not difficult to show that for all  $\psi \in V_\Omega$ , the evaluation  $\psi(\xi) = 0$  for all  $\xi \in \Omega$ .

### B. The RKHS Embedding Method

In this paper, when we refer to the classical problem of adaptive estimation for an unknown nonlinear set of ODEs, we assume that the state trajectory  $t \mapsto x(t) \in \mathbb{R}^d$  satisfies the set of equations

$$\dot{x}(t) = Ax(t) + Bf(x(t)) \quad (6)$$

where, as in the error equations,  $A \in \mathbb{R}^{d \times d}$  is a known Hurwitz matrix,  $B \in \mathbb{R}^{d \times 1}$  is known, and  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is unknown and to be identified. Under the assumption that  $f(\cdot) = \sum_{i=1}^n \alpha_i^* \phi_i(\cdot)$  for a set of given regressor functions,  $\phi_i : \mathbb{R}^d \rightarrow \mathbb{R}$ , the choice of estimator and learning law

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + B\Phi^T(x(t))\hat{\alpha}(t) \\ \dot{\hat{\alpha}}(t) &= \mu[B\Phi^T(x(t))]^T P(x(t) - \hat{x}(t)) \end{aligned} \quad (7)$$

induce the error equations in Eq. 1.

In the RKHS embedding formulation, Eq. 6 is interpreted as the functional equation

$$\dot{x}(t) = Ax(t) + B\mathcal{E}_{x(t)}f, \quad (8)$$

and the corresponding estimation equation and learning law are

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + B\mathcal{E}_{x(t)}\hat{f}(t), \\ \dot{\hat{f}}(t) &= \mu(B\mathcal{E}_{x(t)})^* P(x(t) - \hat{x}(t)), \end{aligned} \quad (9)$$

which induce the dynamics of the error in  $\mathbb{R}^d \times H_\Omega$  in terms of Eq. 5.

## III. MAIN RESULTS

*Theorem 1:* The PE condition in Definition 2 implies the one in Definition 3 for all  $g \in H_\Omega$ .

*Proof:* If the condition in Definition 3 holds, there exist constants  $T_0, \gamma, \delta$ , and  $\Delta$ , such that for each  $t \geq T_0$  and any  $g \in H_\Omega$  with  $\|g\|_{H_\Omega} = 1$ , there exists  $s \in [t, t + \Delta]$  such that

$$\left| \int_s^{s+\delta} \mathcal{E}_{x(\tau)} g d\tau \right| \geq \gamma > 0.$$

By the definition of adjoint operator, the integral in Eq. 4 equals

$$\begin{aligned} \int_t^{t+\Delta} \left( \mathcal{E}_{x(\tau)}^* \mathcal{E}_{x(\tau)} g, g \right)_{H_\Omega} d\tau &= \int_t^{t+\Delta} (\mathcal{E}_{x(\tau)} g, \mathcal{E}_{x(\tau)} g) d\tau, \\ &\geq \int_s^{s+\delta} (\mathcal{E}_{x(\tau)} g)^2 d\tau. \end{aligned}$$

It is assumed that  $t \mapsto x(t)$  is continuous, and  $g : x \mapsto g(x)$  is continuous since  $H_\Omega \hookrightarrow \mathcal{C}(X)$ . Hence,  $g \circ x$  is also a continuous mapping. Moreover, the interval  $[t, t + \Delta]$  is compact, so  $g(x(t))$  is bounded in this interval. Therefore, the function  $\mathcal{E}_{x(\cdot)} g = g(x(\cdot)) \in \mathcal{L}^2([t, t + \Delta], \mathbb{R})$ , and the same applies to the integrand  $(\mathcal{E}_{x(t)} g)^2$ . By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_s^{s+\delta} 1 d\tau \int_s^{s+\delta} (\mathcal{E}_{x(\tau)} g)^2 d\tau &\geq \left( \int_s^{s+\delta} |\mathcal{E}_{x(\tau)} g| d\tau \right)^2, \\ &\geq \left| \int_s^{s+\delta} \mathcal{E}_{x(\tau)} g d\tau \right|^2 \geq \gamma^2. \end{aligned}$$

which then implies

$$\int_t^{t+\Delta} \left( \mathcal{E}_{x(\tau)}^* \mathcal{E}_{x(\tau)} g, g \right)_{H_\Omega} d\tau \geq \gamma^2 / \delta > 0. \quad \blacksquare$$

*Theorem 2:* Suppose that the PE condition in Definition 3 holds and the family of functions  $\mathcal{U}(\bar{S}_1)$  is uniformly equicontinuous. Then the PE condition in Definition 3 (PE.2) implies the one in Definition 2 (PE.1).

*Proof:* Suppose the condition in Definition 3 holds. For each  $t \geq T_0$  and  $g \in H_\Omega$  with  $\|g\|_{H_\Omega} = 1$ , we have

$$\int_t^{t+\Delta} \left( \mathcal{E}_{x(\tau)}^* \mathcal{E}_{x(\tau)} g, g \right)_{H_\Omega} d\tau = \int_t^{t+\Delta} g(x(\tau))^2 d\tau \geq \gamma.$$

By the mean value theorem, there exists a  $\xi \in [t, t + \Delta]$  such that  $\int_t^{t+\Delta} g(x(\tau))^2 d\tau = g(x(\xi))^2 \Delta$ . Thus we have

$$g(x(\xi))^2 \Delta \geq \gamma \Rightarrow |g(x(\xi))| \geq \sqrt{\gamma/\Delta}. \quad (10)$$

For  $\epsilon = \frac{1}{2} \sqrt{\gamma/\Delta}$ , since  $g \in \mathcal{U}(\bar{S}_1)$  and is uniformly equicontinuous, there exist  $\delta = \delta(\epsilon) > 0$  such that  $|s - \xi| < \delta$  implies  $|g(x(s)) - g(x(\xi))| < \epsilon = \frac{1}{2} \sqrt{\gamma/\Delta}$  for all  $g(x(\cdot))$ . In other words, for all  $s \in [\xi - \delta, \xi + \delta]$ , we have

$$|g(x(s)) - g(x(\xi))| < \frac{1}{2} \sqrt{\gamma/\Delta},$$

which implies that

$$|g(x(s))| > |g(x(\xi))| - \frac{1}{2} \sqrt{\gamma/\Delta} \geq \frac{1}{2} \sqrt{\gamma/\Delta}.$$

This implies that  $g(x(s))$  does not change its sign in the interval  $[s, s + \delta]$ . Therefore,

$$\left| \int_s^{s+\delta} \mathcal{E}_{x(\tau)} g d\tau \right| \geq \frac{\delta}{2} \sqrt{\gamma/\Delta} > 0. \quad \blacksquare$$

The following lemma is one intuitive way that the uniform equicontinuity condition can be achieved. It relies on the fact that a uniformly bounded derivative can be used to show a function is Lipschitz continuous [23].

*Lemma 1:* Let  $g \in \bar{S}_1$ , that is,  $g \in H_\Omega$  with  $\|g\|_{H_\Omega} = 1$ . The family of functions  $\mathcal{U}(\bar{S}_1)$  is defined as stated above. Suppose there is a constant  $L > 0$  such that  $\|\partial g(\xi)/\partial \xi\| \leq L$  for all  $x \in \Omega$  and  $g \in \bar{S}_1$ . Then  $\mathcal{U}(\bar{S}_1)$  is uniformly equicontinuous.

*Proof:* The norm of evaluation operator  $\|\mathcal{E}_x\|_{op}$  is uniformly bounded due to the uniform embedding, which then guarantees that  $\|\dot{\tilde{x}}(t)\| \leq c_0$  for all  $t \geq 0$ . Then the lemma is an immediate consequence of Lemma 3.1 in [23]. ■

It is clear that the PE.2 is stronger than PE.1. However, the statement of PE.2 seems better suited for “geometric” interpretations. One established result [18] is that the condition PE.2 can be directly used to relate the indexing set  $\Omega$ , the RKHS  $H_\Omega$ , and the positive orbit  $\Gamma^+(x_0)$  of system in Eq. 6 in an intuitive manner. That is, the positive orbit  $\Gamma^+(x_0)$  persistently excites  $H_\Omega$  implies the indexing set  $\Omega$  is a subset of the  $\omega$ -limit set of  $\Gamma^+(x_0)$ . Readers are referred to see [18] for a detailed discussion.

*Theorem 3:* Assume that the trajectory  $t \mapsto x(t)$  persistently excites the RKHS  $H_\Omega$  in the sense of Definition 2 (PE.1), the initial condition  $\hat{f}(0) \in H_\Omega$ ,  $f \in H_\Omega$ , and  $x(t) \in \Omega$  for all time  $t$ . Then the estimation error system in Eq. 5 is uniformly asymptotically stable at the origin. In particular, we have  $\lim_{t \rightarrow \infty} \|\tilde{x}(t)\| = 0$  and  $\lim_{t \rightarrow \infty} \|\tilde{f}(t)\|_{H_\Omega} = 0$ .

*Proof:* Without loss of generality, we assume that  $\mu = 1$  in Eq. 5. We first establish that  $\tilde{f}(t) \in H_\Omega$ , so that the expression  $\|\tilde{f}(t)\|_{H_\Omega}$  makes sense. Suppose that we integrate the update law in Equations 5 and take the inner product of the result with an element  $g \in V_\Omega$ ,

$$(\hat{f}(t), g)_{H_\Omega} = (\hat{f}(0), g)_{H_\Omega} + \int_0^t \mu(\mathfrak{K}_{x(\tau)}, g)_{H_\Omega} B^T P \tilde{x}(\tau) d\tau.$$

As mentioned in Section II, if  $g \in V_\Omega$ , then  $g(\xi) = 0$  for all  $\xi \in \Omega$ . If the trajectory  $x(\tau) \in \Omega$  for each  $\tau \in [0, t]$ , then the second line above is equal to zero. If in addition, the initial condition  $\hat{f}(0) \in H_\Omega$ , it follows from the fact that  $\tilde{f}(0) \in H_\Omega$  that  $\tilde{f}(t) \in H_\Omega$  for all time  $t$ .

Now we prove  $\lim_{t \rightarrow \infty} \|\tilde{x}(t)\| = 0$ . Consider the candidate Lyapunov function  $V(t) = (\tilde{x}(t), P\tilde{x}(t)) + (\tilde{f}(t), \tilde{f}(t))_{H_X}$ , where  $P \in \mathbb{R}^{d \times d}$  is the unique positive definite solution to the Lyapunov equation  $A^T P + PA = -Q$  for  $Q \succ 0$ . Clearly,  $V(x)$  is bounded below by zero. We take the time derivative of  $V(t)$  along any trajectory of Eq. 5, then apply the Lyapunov equation.

$$\begin{aligned} \dot{V}(t) &= (\tilde{x}(t), (A^T P + PA)\tilde{x}(t)) + 2(B\mathcal{E}_{x(t)}\tilde{f}(t), P\tilde{x}(t)) \\ &\quad + 2(\tilde{f}(t), -(B\mathcal{E}_{x(t)})^* P x)_{H_X} = -(\tilde{x}(t), Q\tilde{x}(t)). \end{aligned}$$

Since  $Q$  is positive definite,  $\dot{V}(t)$  is less than zero, which implies  $V(t)$  is nonincreasing. For all  $t \geq t_0$ ,  $V(t) \leq V(t_0)$ . We conclude that Eq. 5 is stable.

Integrating  $\dot{V}(t)$ , we have the following equation that holds for all  $t \geq t_0$ ,  $\int_{t_0}^t (\tilde{x}(\tau), Q\tilde{x}(\tau)) d\tau = V(t_0) - V(t)$ .

Note that  $V(t_0) < \infty$  and that  $V(t) \geq 0$  is nonincreasing. By sending  $t$  to infinity, we can bound the following improper integral

$$\int_{t_0}^{\infty} (\tilde{x}(\tau), Q\tilde{x}(\tau)) d\tau = V(t_0) - \lim_{t \rightarrow \infty} V(t) < \infty. \quad (11)$$

We claim that the integrand  $(\tilde{x}(t), Q\tilde{x}(t))$  is uniformly continuous with respect to time  $t$ . Then by Barbalat’s lemma [23], [24], it can be deduced from Eq. 11 that  $\lim_{t \rightarrow \infty} (\tilde{x}(t), Q\tilde{x}(t)) = 0$ , which implies

$$\lim_{t \rightarrow \infty} \|\tilde{x}(t)\| = 0. \quad (12)$$

Now we show that  $(\tilde{x}(t), Q\tilde{x}(t))$  is uniformly continuous. Since for all  $t \geq t_0$ , it holds that

$$V(t) = (\tilde{x}(t), P\tilde{x}(t)) + (\tilde{f}(t), \tilde{f}(t))_{H_X} \leq V(t_0) < \infty.$$

It follows that  $\|\tilde{x}(t)\|$  and  $\|\tilde{f}(t)\|_{H_X}$  are both bounded. As stated in Section II, the RKHS  $H_X$  is uniformly embedded in the continuous functions, so  $\|\mathcal{E}_x\|_{op} \leq \bar{k}$ . Thus from Eq. 5, we have

$$\|\dot{\tilde{x}}(t)\| \leq \|A\|_{op} \|\tilde{x}(t)\| + \|B\| \|\mathcal{E}_{x(t)}\|_{op} \|\tilde{f}(t)\|_{H_X} < \infty.$$

Thus  $\|\dot{\tilde{x}}(t)\|$  is uniformly bounded, which further implies that  $\frac{d}{dt}(\tilde{x}(t), Q\tilde{x}(t))$  is bounded uniformly. Therefore,  $(\tilde{x}(t), Q\tilde{x}(t))$  is Lipschitz continuous with respect to  $t$ , which implies the uniform continuity.

According to Eq. 12, for all  $\epsilon > 0$ , there exists  $T$  such that for all  $t \geq T$ ,  $\|\tilde{x}(t)\| < \epsilon$ . Now we consider the PE condition. Let  $g = \tilde{f}(T)/\|\tilde{f}(T)\|_{H_\Omega}$  be the unit-norm function in Eq. 3. If PE condition in Definition 2 is satisfied, there exists  $s \in [T, T + \Delta]$  such that  $\left| \int_s^{s+\delta} \mathcal{E}_{x(\tau)} g d\tau \right| \geq \gamma > 0$ . Consider the error in state  $\tilde{x}(s + \delta)$ . It can be bounded by integrating the state equation in Eq. 5.

$$\begin{aligned} \|\tilde{x}(s + \delta)\| &= \left\| \tilde{x}(s) + \int_s^{s+\delta} A\tilde{x}(\tau) + B\mathcal{E}_{x(\tau)}\tilde{f}(\tau) d\tau \right\|, \\ &\geq \underbrace{\left\| \int_s^{s+\delta} B\mathcal{E}_{x(\tau)}\tilde{f}(T) d\tau \right\|}_{\text{term 1}} - \underbrace{\left\| \tilde{x}(s) + \int_s^{s+\delta} A\tilde{x}(\tau) d\tau \right\|}_{\text{term 2}} \\ &\quad - \underbrace{\left\| \int_s^{s+\delta} B\mathcal{E}_{x(\tau)}(\tilde{f}(\tau) - \tilde{f}(T)) d\tau \right\|}_{\text{term 3}}. \quad (13) \end{aligned}$$

In term 1, note that  $\tilde{f}(T) = g\|\tilde{f}(T)\|_{H_\Omega}$ , and  $\|\tilde{f}(T)\|_{H_\Omega}$  is a constant. The coefficient matrix  $B \in \mathbb{R}^{d \times 1}$  is in fact a  $d$ -dimensional vector, so  $\|B\|_{op} = \|B^*\|_{op} = \|B\|$ . In term 2, note that  $\|\tilde{x}(t)\| < \epsilon$  for all  $t \geq T$ . Then we have

$$\begin{aligned} \text{term 1} &= \left| \int_s^{s+\delta} \mathcal{E}_{x(\tau)} g d\tau \right| \|B\| \|\tilde{f}(T)\|_{H_\Omega}, \\ &\geq \gamma \|\tilde{f}(T)\|_{H_\Omega}. \quad (14) \end{aligned}$$

$$\begin{aligned} \text{term 2} &\leq \|\tilde{x}(s)\| + \int_s^{s+\delta} \|A\|_{op} \|\tilde{x}(\tau)\| d\tau, \\ &\leq \epsilon + \|A\|_{op} \delta \epsilon. \quad (15) \end{aligned}$$

For term 3, we first derive a bound on  $\tilde{f}(\tau) - \tilde{f}(T)$ , which can be obtained by integrating Eq. 5.

$$\begin{aligned} \|\tilde{f}(\tau) - \tilde{f}(T)\|_{H_\Omega} &= \left\| \int_T^\tau (B\mathcal{E}_{x(\xi)})^* P\tilde{x}(\xi) d\xi \right\|_{H_\Omega}, \\ &\leq \int_T^\tau \|B^*\|_{op} \|\mathcal{E}_{x(\xi)}^*\|_{op} \|P\|_{op} \|\tilde{x}(\xi)\| d\xi, \\ &\leq \bar{k}\epsilon(\tau - T) \|B\| \|P\|_{op}. \end{aligned}$$

If we let  $c_1 = \|B\| \|P\|_{op}$ , then  $\|\tilde{f}(\tau) - \tilde{f}(T)\|_{H_\Omega} \leq c_1 \bar{k}\epsilon(\tau - T)$ . In term 3, note that  $T \leq s \leq T + \Delta$ . This means

$$\begin{aligned} \text{term 3} &\leq \int_s^{s+\delta} \|B\| \|\mathcal{E}_{x(\tau)}\|_{op} \|\tilde{f}(\tau) - \tilde{f}(T)\|_{H_\Omega} d\tau \quad (16) \\ &\leq \bar{k} \|B\| \int_s^{s+\delta} c_1 \bar{k}\epsilon(\tau - T) d\tau, \leq c_1 \bar{k}^2 \epsilon \|B\| \left(\frac{1}{2}\delta^2 + \Delta\delta\right). \end{aligned}$$

Let  $c_2 = \|B\| \bar{k}^2 (\frac{1}{2}\delta^2 + \Delta\delta)$ . Then term 3  $\leq c_1 c_2 \epsilon$ . Substituting Eq. 14-16 into Eq. 13 gives a lower bound of  $\tilde{x}(s + \delta)$ ,

$$\|\tilde{x}(s + \delta)\| \geq \gamma \|\tilde{f}(T)\|_{H_\Omega} - (1 + \|A\|_{op}\delta)\epsilon - c_1 c_2 \epsilon. \quad (17)$$

On the other hand,  $s + \delta \geq T$ , so we have  $\|\tilde{x}(s + \delta)\| < \epsilon$ . Thus an upper bound on  $\|\tilde{f}(T)\|_{H_\Omega}$  can be derived from Eq. 17 as follows.

$$\|\tilde{f}(T)\|_{H_\Omega} < \frac{\epsilon}{\gamma} ((2 + \|A\|_{op}\delta) + c_1 c_2). \quad (18)$$

Now we have shown that  $\tilde{f}(T)$  is  $\mathcal{O}(\epsilon)$  for some  $T$  that depends on  $\epsilon$ . However, it follows from this that  $\tilde{f}(T')$  is  $\mathcal{O}(\epsilon)$  for all  $T' \geq T$ . To see why this is so, choose any  $T' > T$ . It is still true that  $\|\tilde{x}(t)\| < \epsilon$  for all  $\tau \geq T' > T$ . We can repeat all of the steps above for  $\tau \geq T'$  to conclude that  $\|\tilde{f}(T')\|_{H_\Omega}$  is  $\mathcal{O}(\epsilon)$ . From this we eventually conclude that  $\lim_{t \rightarrow \infty} \|\tilde{f}(t)\|_{H_\Omega} = 0$ . Therefore, the system in Eq. 5 is uniformly asymptotically stable. ■

#### IV. NUMERICAL SIMULATION

This paper studies the DPS defined by the error Equations 5, which are infinite-dimensional. Practical implementation requires finite-dimensional approximations, a careful treatment of which exceeds the limits of this paper. See [15], [16] for some preliminary discussions of the theory of approximations. In this section, we study the qualitative behavior of finite-dimensional approximations, since these are suggestive of the limiting guarantees of this paper. In particular, the analysis of the RKHS embedding method gives additional insights that have no counterpart in the usual finite-dimensional framework.

In particular, the results of this paper can be combined with those in [18]. Reference [18] shows that, with a judicious choice of the kernel  $\mathfrak{K}_X$ , a persistently excited index set is contained in the positive limits set of the original system. This suggests that that one logical choice of a reasonable finite-dimensional approximation can be based on the bases  $\{\mathfrak{K}_{X,z_j}\}_{j=1}^n$  located at the centers  $\Omega_n := \{z_j\}_{j=1}^n$  that are assumed to constitute a good sampling of the positive limit set  $\omega^+(x_0)$  of the orbit  $\Gamma^+(x_0) := \bigcup_{\tau \geq 0} x(\tau)$ . In this way,

we seek estimates that converge in  $H_\Omega$ , that is, they converge over the indexing set  $\Omega_n \subseteq \Omega \equiv \omega^+(x_0)$ .

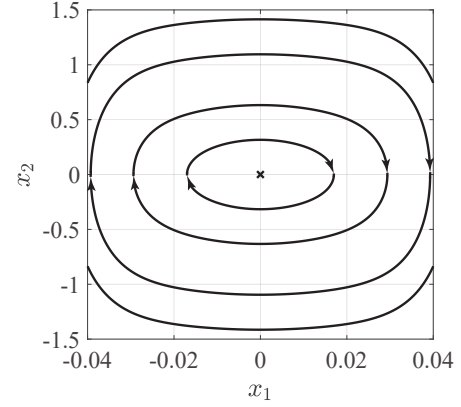


Fig. 1. Phase portrait of the original system

To illustrate the convergence of RKHS embedding method, an example of an undamped, nonlinear, piezoelectric oscillator is studied [25], [26]. The governing equations of the oscillator, after a single bending mode approximation, have the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{k_{n,1}}{m}x_1^3 - \frac{k_{n,2}}{m}x_1^5 \end{bmatrix}, \quad (19)$$

where  $k$  is the electromechanical stiffness,  $m$  is the mass, and  $k_{n,1}, k_{n,2}$  are the higher order electromechanical stiffness coefficients. In the governing equations above, we assume all the linear terms are known, and the nonlinear term  $f(x) = -\frac{k_{n,1}}{m}x_1^3 - \frac{k_{n,2}}{m}x_1^5$  is to be identified. In this case, the Sobolev-Matern kernel is applied, the RKHS associated with which is uniformly embedded in the space of continuous functions [21]. From the conclusion of [18], a persistently excited set  $\Omega$  must be contained in the positive limit set of the system  $\omega^+(x_0)$ , which we approximate by the centers  $\Omega_n = \{z_j\}_{j=1}^n$ .

Fig. 1 shows the typical positive limit sets of this system. The limit sets form limit cycles around the equilibrium at the origin, which is prototypical for such conservative electromechanical oscillators.

When the approximation of infinite dimensional adaptive estimator based on the RKHS embedding technique is implemented for this problem, estimates of the unknown nonlinear function  $f(x)$  are obtained in  $H_{\Omega_n} = \text{span}\{\mathfrak{K}_{X,z_j} : j = 1, \dots, n\}$ . Fig. 2 shows the error between the actual function and function estimate over the state space. Qualitatively, convergence of the function estimate occurs over the positive limit set of a particular trajectory. Fig. 3 shows the contour of the function error along with the positive limit set. Both the figures show that the function estimate in  $H_\Omega$  converges to the actual function over the indexing set, which when persistently excited is a subset of the positive limit set.

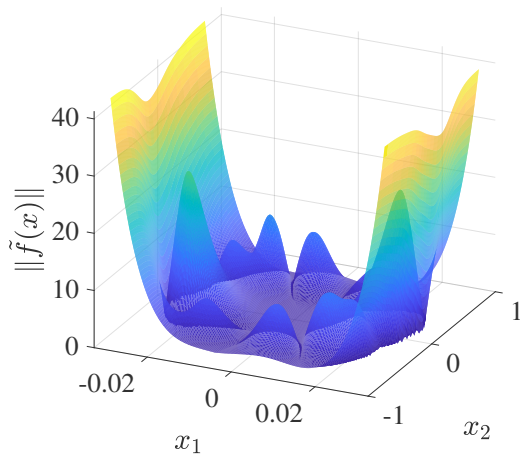


Fig. 2. Error in function estimate

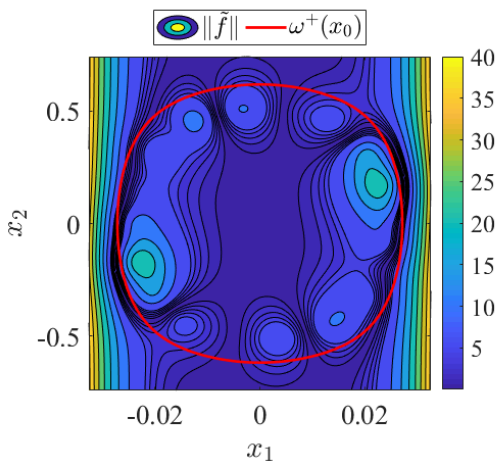


Fig. 3. Error contour

## V. CONCLUSIONS

In this paper, two definitions of PE for the adaptive estimator based on RKHS embedding are given for different purposes, both applied to the family of functions  $\mathcal{U}(\bar{S}_1) = \{g \circ x(\cdot) : g \in H_\Omega, \|g\|_{H_\Omega} = 1\}$ . The paper establishes the equivalence conditions for the two conditions. Condition PE.1 naturally implies PE.2, and PE.2 implies PE.1 when the family of functions  $\mathcal{U}(\bar{S}_1)$  is uniformly equicontinuous. The paper then proves that PE.1 is a sufficient condition for the UAS of the error equations that arise in the RKHS embedding framework. This constitutes a sufficient condition for the convergence of function estimates. A numerical example is given to show qualitatively the convergence behavior of the RKHS embedding method.

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