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THERMODYNAMIC VARIATIONAL FORMULATIONS OF SUBORDINATE OSCILLATOR ARRAYS (SOA) WITH LINEAR PIEZOELECTRICS

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ABSTRACT

It has been shown theoretically that by prescribing the mass and stiffness distributions of a subordinate oscillator array (SOA) that is attached to a host structure, significant vibration attenuation of a host can be obtained over a finite frequency range. This case stands in stark contrast to classical vibration isolator designs for two degree of freedom systems that achieve exact vibration cancellation at a single isolated frequency. Despite the attractiveness of SOAs for the design of broader band vibration suppression, the theoretically desired result can deteriorate rapidly due to small fabrication imperfections in the SOA. This paper introduces and compares variational thermodynamic formulations of composite piezoelectric SOA that are designed to be adjustable in real-time to ameliorate the effects of disorder due to fabrication in a SOA.

INTRODUCTION

Previous work by the authors and other researchers in [14, 6, 1, 2] have studied vibration attenuation methods for a primary structure that are based on attaching to it an array of substructures or appendages. An iconic example of such a finite dimensional, multi-degree of freedom system is depicted in Figure 1. Systems of this type have been referred to as host structures equipped with subordinate oscillators arrays, or SOAs. In effect, the strategy of such an approach is to design the physical properties of the attached structures to rapidly transfer vibrational energy from the host to the SOA. It has been shown that a judicious choice of the mass or stiffness distributions of the SOA can result in vibration attenuation in the host that is characterized by a relatively flat frequency response over a range of excitation frequencies. Such an example response is depicted in Figure 2 from [14, 16]. It is perhaps suprising that this performance can be achieved, in principle, with SOAs that have a total mass that is relatively small compared to the mass of the host structure. Unfortunately, analytic predictions of vibration attenuation featuring such a relatively flat host frequency response in Figure 2 can be quite sensitive to perturbations in the structural properties of the host or SOA. Figure 3 illustrates the effect of introducing disorder on the frequency response function of the host.

One of the implications of these observations is that the structural properties of the host and SOA must be known precisely to achieve theoretical assurances of the performance in practice. Even if the structural properties of the host are known exactly, the fabrication of SOAs must conform closely to design

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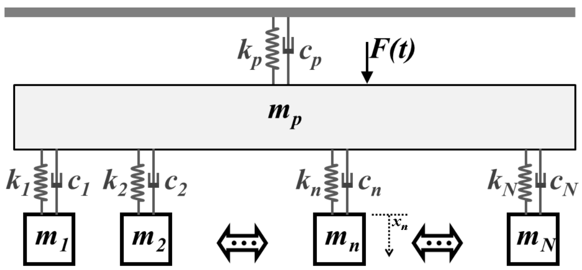

FIGURE 1: A host structure with a subordinate oscillator array (SOA): a finite dimensional, multi-degree of freedom system from [14, 16].

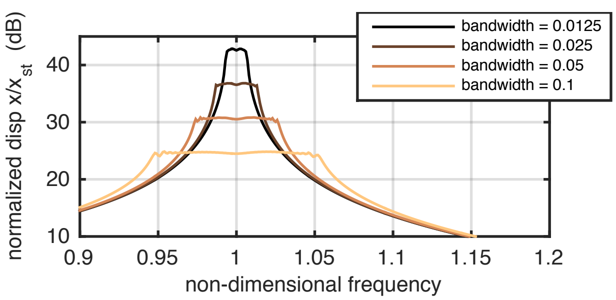

FIGURE 2: The distribution of mass and stiffness properties are designed to obtain vibration suppression over a finite bandwidth from [14, 16].

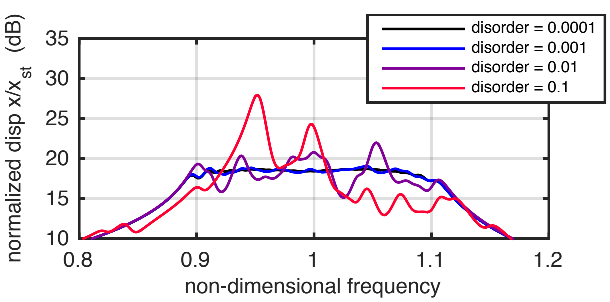

FIGURE 3: The effects of disorder or fabrication errors induce a deterioration of analytic predictions of vibration attenuation from [14, 16].

specifications. Therefore, it would be highly advantageous to be able construct an SOA that is an active system, one that can change its structural properties to account for introduced fabrication errors.

In this paper we study detailed modeling of piezoelectrically

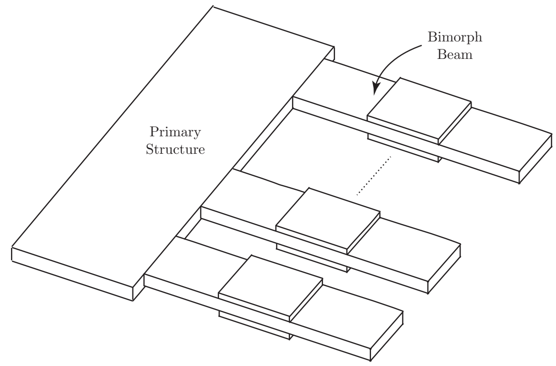

FIGURE 4: The host structure with piezoelectrically actuated SOAs: a distributed parameter system

actuated SOAs as depicted in Figure 4. We begin by reviewing variational formulations expressed in terms of the electric enthalpy density $\mathscr{H}$ for linear piezostructural systems that are coupled to an ideal electrical network. We show how such a formulation can be understood as a generalization of the displacement and flux linkage formulation for electromechanical systems made popular following references such as [5], or in [13, 18, 17]. We introduce a novel complementary variational principle that is expressed in terms of the internal energy density $\mathscr{U}$, displacement, and charge. The complementary variational principle can be viewed as a generalization of the displacment and charge formulation in [5] for finite dimensional electromechanical systems to piezoelectric continua that are connected to an ideal electrical network.

THERMODYNAMIC VARIATIONAL PRINCIPLES

Methods for constructing models of linear and nonlinear piezoelectric composites have a long history and different formulations have appeared in the literature over the years. Early efforts such as in [13] are restricted to consideration of linear piezoelectric continua, while more recent efforts such as in [18] or [17] summarize the relevant theory for nonlinear piezoelectric systems. Roughly speaking, the approaches in these representative studies, and similar ones such as in [17], introduce a variational principle that modifies Hamilton's principle from classical mechanics with one that is cast in terms of the electric enthalpy density $\mathscr{H}$. These references do provide an electric enthalpy density based variational principle for piezoelectric continua coupled to an ideal electrical network.

A related but different viewpoint follows from the early approaches summarized in [5]. The approach in [5] is significant in that it formulates quite general methods for deriving the equations of motion of coupled, finite dimensional electromechanical systems. From first principles, it is shown that electromechanical systems that evolve in a finite dimensional state space can be modeled using complementary charge or voltage variational methods. While [5] focuses on general finite dimensional electromechanical systems, it also includes a chapter on distributed parameter systems, i.e., ones that have an infinite dimensional state space. Still, reference [5] does not discuss a variational formulation for a continuum of piezoelectric material coupled to an ideal electrical network. The culmination of the approaches in [5] yields Lagrange's equations for finite dimensional electromechanical systems in terms of the displacement and charge formulation, or the displacment and flux linkage formulation.

These two formulations are also summarized in [11] in Section 3.4. [5, 11] Reference [11] goes still further, however, and introduces a modification of Hamilton's principle to linear piezoelectric continua in Chapter 4.8. that is cast in terms of the displacement and flux linkage. It is noted in [11], however, that the variational problem is in fact formulated in terms of the electric enthalpy density. Hence, the variational approach in [11] is identical to that in [13, 18, 17]. With a minor modification, the approach derived in [11] is applicable to piezoelectric continua coupled to ideal electrical networks.

Still, other formulations such as in [8] derive finite dimensional governing equations in Chapter 5 for linear piezoelectric continua in terms of the internal energy $\mathscr{U}$. The relationship of the equations of motion generated by approximations of the internal energy such as in Chapter 5 of [8] to the variational methods [13], [18], [17], or [11] cast in terms of the electric enthalpy density $\mathscr{H}$ is not addressed in any of these references.

In the next few sections we will introduce a variational formulation based the internal energy $\mathscr{U}$ of a linear piezoelectric that is coupled to an ideal electrical network. It can be understood as a generalization of the complementary variational statement in the sense of [5, 11] that is cast in terms of displacement and charge for finite dimensional electromechanical systems.

When we introduce the complementary thermodynamic variational principles for linear piezoelectricity, they will be based on modifications of the classical form of Hamilton's principle for nonconservative systems. Recall that the classical form of Hamilton's principle [10] states that the actual trajectory of a mechanical system satisfies the variational identity

$$\delta \int_{t_0}^{t_1} (T - \mathscr{V})dt + \int_{t_0}^{t_1} \delta W_{nc} dt = 0$$

for all admissible variations of the actual trajectory in mechanical configuration space. In this equation T is the kinetic energy, $\mathscr{V}$ is the mechanical potential energy and δW_{nc} is the virtual work performed by any nonconservative mechanical forces acting on the mechanical system.

The $\mathscr{H}$-Variational Principle

We now consider the first thermodynamic variational principle for a linear piezoelectric continua Ω that is coupled to an ideal electrical network. It is expressed in terms of the electric enthalpy density $\mathscr{H}$. This is essentially the variational principle summarized in Section 4.8 of [11], with a minor modification. This principle is expressed in terms of a thermodynamic potential $\mathscr{V}_{\mathscr{H}}$ that depends on the electric enthalpy density $\mathscr{H}$, the displacements, and the flux linkage variables λ_i. The actual trajectory of the electromechanical system satisfies the variational identity

$$\delta \int_{t_0}^{t_1} (T - \mathscr{V}_{\mathscr{H}})dt + \int_{t_0}^{t_1} \delta W_{\mathscr{H},nc} dt = 0$$

for all admissible variations of the actual trajectory in electromechanical configuration space where T is the kinetic energy, $\mathscr{V}_{\mathscr{H}} := \int_\Omega \mathscr{H} d\Omega - \frac{1}{2}\sum_i C_i \dot{\lambda}_i^2$, and $\delta W_{\mathscr{H},nc} = \delta W_{nc} + \sum_k I_k \delta\lambda_k$ is the virtual work of the nonconservative electromechanical loads acting on the system. In this equation I_k is the generalized current associated with the variation in the flux linkage $\delta\lambda_k$. See Section 4.8 of [11] for details regarding the electromechanical virtual work $\delta W_{\mathscr{H},nc}$.

The $\mathscr{U}$-Variational Principle

We next introduce the second thermodynamic variational principle for a linear piezoelectric continua Ω that is coupled to an ideal electrical network. This principle is in terms of the thermodynamic potential $\mathscr{V}_{\mathscr{U}}$ that depends on the internal energy density $\mathscr{U}$, the mechanical displacements, and the charges Q_k. The actual trajectory of the electromechanical systems satisfies the variational identity

$$\delta \int_{t_0}^{t_1} (T - \mathscr{V}_{\mathscr{U}})dt + \int_{t_0}^{t_1} \delta W_{\mathscr{U},nc} dt = 0$$

for all admissible variations of the actual trajectory in electromechanical configuration space where T is the kinetic energy, $\mathscr{V}_{\mathscr{U}} := \int_\Omega \mathscr{U} d\Omega + \frac{1}{2}\sum_i \frac{1}{C_i} Q_i^2$, and $\delta W_{\mathscr{U},nc} = \delta W_{nc} + \sum_k \lambda_k \delta Q_k$ is the virtual work of the nonconservative electromechanical loads acting on the system.

A Prototypical Comparison

We will see that these two thermodynamic variational formulations are equivalent as illustrated in an iconic example. Consider the piezoelectric composite beam depicted in Figure 5. We analyze this system here to demonstrate the equivalence of the two variational formulations, and we will then generate the governing equations of an SOA whose appendages have the geometry based on Figure 4. We first apply the thermodynamic varia-

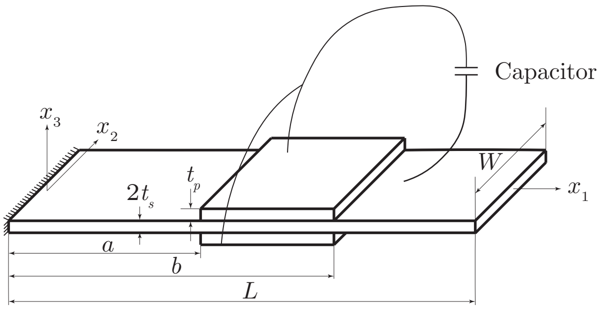

FIGURE 5: Piezoelectric Composite Beam Coupled to an Ideal Electrical Network

tional principle expressed in terms of $\mathscr{V}_{\mathscr{U}}$, the displacements, and the charge. The internal energy density $\mathscr{U}$ is derived in either [7] or [8], and we see that its approximation for a one dimensional domain can be used to express $\mathscr{V}_{\mathscr{U}}$ as

$$\begin{aligned}\mathscr{V}_{\mathscr{U}} &= \int_{\Omega} \mathscr{U}\, dv + \frac{1}{2C} Q_3^2 \\ &= \int_{\Omega} \left\{ \frac{1}{2} C_{11}^D S_{11}^2 + d_{31} S_{11} D_3 + \frac{1}{2} \beta_{33}^S D_3^2 \right\} dv + \frac{1}{2C} Q_3^2.\end{aligned}$$

Substituting the strain displacement relationship $S_{11} = -x_3 \frac{\partial^2 w}{\partial x_1^2}$, the piezoelectric patch surface area $A_p = (b-a) \times w$ and cross sectional area $A = t_p \times w$, and the electric displacement

$$D_3 = \begin{cases} -\frac{Q_3}{A_p} \chi[a,b](x_1) & (x_1, x_2, x_3) \in \text{ top patch}, \\ \frac{Q_3}{A_p} \chi[a,b](x_1) & (x_1, x_2, x_3) \in \text{ bottom patch}, \\ 0 & \text{otherwise}, \end{cases}$$

we see that

$$\begin{aligned}\mathscr{V}_{\mathscr{U}} &= \int_{\Omega} \left\{ \frac{1}{2} \left(x_3 \frac{\partial^2 w}{\partial x_1^2} \right)^2 C_{11}^D + \left(x_3 \frac{\partial^2 w}{\partial x_1^2} \right) \frac{d_{31} Q_3 \chi_{[a,b]}}{A_p} \right. \\ &\qquad \left. + \frac{1}{2} \left(\frac{Q_3}{A_p} \chi_{[a,b]} \right)^2 \beta_{33}^S \right\} dv + \frac{1}{2C} Q_3^2, \\ &= \int_0^L \left\{ \frac{1}{2} C_{11}^D I \left(\frac{\partial^2 w}{\partial x_1^2} \right)^2 + \left(\frac{\partial^2 w}{\partial x_1^2} \right) \frac{K d_{31} Q_3 \chi_{[a,b]}}{A_p} \right. \\ &\qquad \left. + \frac{1}{2} \frac{Q_3^2}{A_p^2} A \beta_{33}^S \chi_{[a,b]} \right\} dx_1 + \frac{1}{2C} Q_3^2.\end{aligned}$$

The kinetic energy in this example problem is written as

$$T = \frac{1}{2} \int_0^L \rho_m A \left(\frac{\partial w}{\partial t} \right)^2 dx_1.$$

The modified Hamilton's principle of piezoelectricty in terms of displacements and charges requires that

$$\delta \int_{t_0}^{t_1} (T - \mathscr{V}_{\mathscr{U}})\, dt + \int_{t_0}^{t_1} \delta W_{\mathscr{U},nc} dt = 0$$

for all admissible variations of the true trajectory in electromechanical configuration space. Following integration by parts and enforcement of the boundary conditions on the variations in electromechanical configuration space, we find that we must have

$$\begin{aligned}&\int_{t_0}^{t_1} \int_0^L \left(-\rho_m A \left(\frac{\partial^2 w}{\partial t^2} \right) - \frac{\partial^2}{\partial x_1^2} \left(C_{11}^D I \left(\frac{\partial^2 w}{\partial x_1^2} \right) \right) \right. \\ &\qquad \left. - \frac{\partial^2}{\partial x_1^2} \left(\frac{K d_{31} \chi_{[a,b]}}{A_p} \right) Q_3 \right) \delta w dx_1 dt \\ &\qquad - \int_{t_0}^{t_1} \left(\int_0^L \left(\frac{K d_{31} \chi_{[a,b]}}{A_p} \left(\frac{\partial^2 w}{\partial x_1^2} \right) - \frac{Q_3}{A_p^2} A \beta_{33}^S \chi_{[a,b]} \right) dx_1 \right. \\ &\qquad \left. - \frac{1}{C} Q_3 \right) \delta Q_3 dt \\ &\qquad + \text{ variational BCs} = 0\end{aligned}$$

for all admissible variations δw and δQ_3. We find that the strong form of a solution (w, Q_3) must satisfy the pair of equations

$$\begin{aligned}\rho_m A \left(\frac{\partial^2 w}{\partial t^2} \right) + \frac{\partial^2}{\partial x_1^2} \left(C_{11}^D I \left(\frac{\partial^2 w}{\partial x_1^2} \right) \right) \\ + \frac{\partial^2}{\partial x_1^2} \left(\frac{K d_{31} Q_3 \chi_{[a,b]}}{A_p} \right) = 0, \qquad (1)\end{aligned}$$

$$-\int_0^L \frac{K d_{31} \chi_{[a,b]}}{A_p} \left(\frac{\partial^2 w}{\partial x_1^2} \right) dx_1 + \frac{Q_3}{A_p^2} A \beta_{33}^S (b-a) + \frac{1}{C} Q_3 = 0, \qquad (2)$$

for all $(t, x_3) \in [0, \infty) \times [0, L]$, subject to initial conditions and to appropriate variational boundary conditions.

We next consider the thermodynamical variational principle that is expressed in terms of $\mathscr{V}_{\mathscr{H}}$, the displacements $w = w(t,x)$, and flux linkage variables λ. We write the electric enthalpy density in the form [7], [13]

$$\mathscr{H} := \frac{1}{2} C_{11}^E S_{11}^2 - e_{31} S_{11} E_3 - \frac{1}{2} \varepsilon_{33}^S E_3^2,$$

so that

$$\mathscr{V}_{\mathscr{H}} = \frac{1}{2}\int_0^L C_{11}^E \left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 dx_1 - \int_0^L \frac{Ke_{31}\chi_{[a,b]}}{t_p} V \frac{\partial^2 w}{\partial x_1^2} dx_1 - \frac{1}{2}\int_0^L A_p \varepsilon_{33}^S \chi_{[a,b]} \left(\frac{V}{t_p}\right)^2 dx_1 - \frac{1}{2} C V^2.$$

The modified Hamilton's principle of piezoelectricity in terms of displacements and flux linkage imposes the variational statement that

$$\delta \int_{t_0}^{t_1} (T - \mathscr{V}_{\mathscr{H}})\, dt + \int_{t_0}^{t_1} \delta W_{\mathscr{H},nc} dt = 0$$

for all admissible variations of the actual trajectory in electromechanical configuration space. After using standard tools from variational calculus, we find that the equation

$$\begin{aligned}&\int_{t_0}^{t_1}\int_0^L \left\{ -\rho_m A \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial x_1^2}\left(C_{11}^E I \frac{\partial^2 w}{\partial x_1^2}\right) + \frac{\partial^2}{\partial x_1^2}\left(\frac{Ke_{31}\chi_{[a,b]}}{t_p} V\right)\right\} \delta w dx_1 dt \\ &+ \int_{t_0}^{t_1}\left\{\int_0^L \frac{Ke_{31}\chi_{[a,b]}}{t_p}\frac{\partial^2 w}{\partial x_1^2} dx_1 + \frac{A_p \varepsilon_{33}^S (b-a)}{t_p^2} V + CV\right\} \delta V dt + \text{ variational BCs} = 0\end{aligned}$$

must hold for all admissible variations δw and $\delta\lambda$ of the actual trajectory in electromechanical configuration space. A strong solution (w,λ) of the governing equations must satisfy

$$\rho_m A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x_1^2}\left(C_{11}^E I \frac{\partial^2 w}{\partial x_1^2}\right) - \frac{\partial^2}{\partial x_1^2}\left(\frac{Ke_{31}\chi_{[a,b]}}{t_p}\right) V = 0, \quad (3)$$

$$\int_0^L \frac{Ke_{31}\chi_{[a,b]}}{t_p}\frac{\partial^2 w}{\partial x_1^2} dx_1 + \frac{A_e \varepsilon_{33}^S (b-a)}{t_p^2} V + CV = 0, \quad (4)$$

for all $(t,x_3) \in [0,\infty) \times [0,L]$, subject to initial conditions and to appropriate variational boundary conditions.

Substituting the constitutive relationship $C_{11}^E = C_{11}^D + d_{31}e_{31}$ and a bit of manipulation shows that Equations 1, 2 are equivalent to Equations 3,4. For example, starting with the equation of motion derived from the variational formulation that is expressed in terms of $\mathscr{V}_{\mathscr{H}}$, the displacements, and the flux linkage, we obtain

$$\begin{aligned}0 &= \rho_m A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x_1^2}\left((C_{11}^D + d_{31}e_{31}) I \frac{\partial^2 w}{\partial x_1^2}\right) - \frac{\partial^2}{\partial x_1^2}\left(\frac{Ke_{31}\chi_{[a,b]}}{t_p}\right) V, \\ &= \rho_m A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x_1^2}\left(C_{11}^D I \frac{\partial^2 w}{\partial x_1^2}\right) \\ &+ \frac{\partial^2}{\partial x_1^2}\left(d_{31}e_{31}\iint x_3^2 dx_2 dx_3 \frac{\partial^2 w}{\partial x_1^2} - \iint x_3 dx_2 dx_3 \frac{e_{31}\chi_{[a,b]}}{t_p} V\right), \\ &= \rho_m A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x_1^2}\left(C_{11}^D I \frac{\partial^2 w}{\partial x_1^2}\right) \\ &- d_{31}\frac{\partial^2}{\partial x_1^2}\iint (e_{31}S_{11} + \varepsilon_{33}E_3) x_3 dx_2 dx_3, \\ &= \rho_m A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x_1^2}\left(C_{11}^D I \frac{\partial^2 w}{\partial x_1^2}\right) - d_{31}\frac{\partial^2}{\partial x_1^2}\iint D_3 x_3 dx_2 dx_3, \\ &= \rho_m A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x_1^2}\left(C_{11}^D I \frac{\partial^2 w}{\partial x_1^2}\right) + \frac{\partial^2}{\partial x_1^2}\left(d_{31}K\frac{Q_3}{A_p}\chi[a,b]\right).\end{aligned}$$

These computations have shown that Equation 3 is equivalent to Equation 1. A similar analysis shows that Equation 4 is equivalent to Equation 2.

PIEZOELECTRIC SOA MODEL

We now use the approaches outlined in this paper to construct an approximation of the piezoelectrically actuated SOA depicted in Figure 4. Each appendage in Figure 4 has the geometry as depicted in Figure 5. The total kinetic energy for a base-driven primary structure is written in the form

$$T = \frac{1}{2} m_p (\dot{x}_p - \dot{z})^2 + \sum_{i=1}^N \left\{\frac{1}{2}\int_0^{L_i} \rho_i A_i \left(\dot{x}_p + \frac{\partial w_i}{\partial t} - \dot{z}\right)^2 dx_i + \frac{1}{2} m_i \left(\dot{x}_p + \frac{\partial w_i}{\partial t}(t,L_i) - \dot{z}\right)^2\right\}.$$

with x_p the displacement of the primary structure, $w_i := w_i(t,x_i)$ the displacement along appendage i at the location x_i, m_i the top mass of the i^{th} appendage, L_i the length of the i^{th} appendage, ρ_i the mass density of the i^{th} appendage, and A_i the cross sectional area of the i^{th} appendage. We construct Galerkin approximations of the transverse displacement of the i^{th} appendage in the form $w_i(t,x_i) = \sum_{k=1}^{n_i} \psi_{i,k}(x_i) w_{i,k}(t) = \boldsymbol{\Psi}_i^T(x_i)\boldsymbol{W}_i(t) = \boldsymbol{W}_i^T(t)\boldsymbol{\Psi}_i(x_i)$ for $i = 1,\ldots,n_i$, with the vectors $\boldsymbol{\Psi}_i$ and $\boldsymbol{W}_i$ defined as $\boldsymbol{\Psi}_i := \left\{\Psi_{i,1} \cdots \Psi_{i,n_i}\right\}^T$ and $\boldsymbol{W}_i := \left\{W_{i,1} \cdots W_{i,n_i}\right\}^T$. The contribution of the distributed mass of the i^{th} appendage to the kinetic energy is then expressed as

$$\begin{aligned}T_i &:= \frac{1}{2}\int_0^{L_i} \rho_i A_i \left((\dot{x}_p-\dot{z})+\dot{\boldsymbol{W}}_i^T\boldsymbol{\Psi}_i\right)\left((\dot{x}_p-\dot{z})+\boldsymbol{\Psi}_i^T\dot{\boldsymbol{W}}_i\right)dx_i,\\ &= \frac{1}{2}\Bigg(\underbrace{\int_0^{L_i}\rho_i A_i dx_i}_{\mathscr{M}_i}(\dot{x}_p-\dot{z})^2+2\underbrace{\int_0^{L_i}\rho_i A_i \boldsymbol{\Psi}_i^T dx_i}_{\mathscr{M}_{ip}^T}\dot{\boldsymbol{W}}_i(\dot{x}_p-\dot{z})\\ &\quad +\dot{\boldsymbol{W}}_i^T\underbrace{\int_0^{L_i}\rho_i A_i\boldsymbol{\Psi}_i\boldsymbol{\Psi}_i^T dx_i}_{\mathscr{M}_{ii}}\dot{\boldsymbol{W}}_i\Bigg).\end{aligned}$$

This expression then reduces to

$$T_i = \frac{1}{2}\left(\mathscr{M}_i(\dot{x}_p-\dot{z})^2+2\mathscr{M}_{ip}^T\dot{\boldsymbol{W}}_i(\dot{x}_p-\dot{z})+\dot{\boldsymbol{W}}_i^T\mathscr{M}_{ii}\dot{\boldsymbol{W}}_i\right).$$

Contribution of the tip masses of each appendage to the kinetic energy is calculated similarly. We have

$$\begin{aligned}T_i &= \frac{1}{2}m_i\left((\dot{x}_p-\dot{z})+\dot{\boldsymbol{W}}_i^T\boldsymbol{\Psi}_i(L_i)\right)\left((\dot{x}_p-\dot{z})+\boldsymbol{\Psi}_i^T(L_i)\dot{\boldsymbol{W}}_i\right),\\ &= \frac{1}{2}\Bigg(m_i(\dot{x}_p-\dot{z})^2+2\underbrace{m_i\boldsymbol{\Psi}_i^T(L_i)}_{\boldsymbol{m}_{ip}^T}\dot{\boldsymbol{W}}_i(\dot{x}_p-\dot{z})\\ &\quad +\dot{\boldsymbol{W}}_i^T\underbrace{m_i\boldsymbol{\Psi}_i(L_i)\boldsymbol{\Psi}_i^T(L_i)}_{\boldsymbol{m}_{ii}}\dot{\boldsymbol{W}}_i\Bigg),\\ &= \frac{1}{2}\left(m_i(\dot{x}_p-\dot{z})^2+2\boldsymbol{m}_{ip}^T\dot{\boldsymbol{W}}_i(\dot{x}_p-\dot{z})+\dot{\boldsymbol{W}}_i^T\boldsymbol{m}_{ii}\boldsymbol{W}_i\right).\end{aligned}$$

We sum the above contributions to obtain the total kinetic energy

$$\begin{aligned}T &= \frac{1}{2}m_p(\dot{x}_p-\dot{z})^2+\sum_{i=1}^N\frac{1}{2}\Bigg(\underbrace{(\mathscr{M}_i+m_i)}_{M_i}(\dot{x}_p-\dot{z})^2\\ &\quad +2\underbrace{(\mathscr{M}_{ip}+\boldsymbol{m}_{ip})}_{M_{ip}}{}^T\dot{\boldsymbol{W}}_i^T(\dot{x}_p-\dot{z})+\dot{\boldsymbol{W}}^T\underbrace{(\mathscr{M}_{ii}+\boldsymbol{m}_{ii})}_{M_{ii}}\dot{\boldsymbol{W}}\Bigg),\\ &= \frac{1}{2}m_p(\dot{x}_p-\dot{z})^2\\ &\quad +\sum_{i=1}^N\frac{1}{2}\left(M_i(\dot{x}_p-\dot{z})^2+2M_{ip}^T\dot{\boldsymbol{W}}_i(\dot{x}_p-\dot{z})+\dot{\boldsymbol{W}}^T M_{ii}\dot{\boldsymbol{W}}_i\right),\\ &= \frac{1}{2}M_{pp}(\dot{x}_p-\dot{z})^2+\sum_{i=1}^N\frac{1}{2}\left(2M_{ip}^T\dot{\boldsymbol{W}}_i(\dot{x}_p-\dot{z})+\dot{\boldsymbol{W}}^T M_{ii}\dot{\boldsymbol{W}}_i\right),\end{aligned}$$

where $M_{pp} := m_p+\sum_{i=1}^N M_i$. The total kinetic energy can be expressed in the quadratic form. We have,

$$T = \frac{1}{2}\left\{\dot{\boldsymbol{W}}_1^T\ \cdots\ \dot{\boldsymbol{W}}_n^T \middle| (\dot{x}_p-\dot{z})\right\} \times \left[\begin{array}{cccc|c} \boldsymbol{M}_{11} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{M}_{1p}\\ \boldsymbol{0} & \boldsymbol{M}_{22} & \boldsymbol{0} & \cdots & \boldsymbol{M}_{2p}\\ \vdots & \boldsymbol{0} & \ddots & \boldsymbol{0} & \vdots\\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{M}_{nn} & \boldsymbol{M}_{np}\\ \hline \boldsymbol{M}_{1p}^T & \boldsymbol{M}_{2p}^T & \cdots & \boldsymbol{M}_{np}^T & M_{pp}\end{array}\right] \times \left\{\begin{array}{c}\dot{\boldsymbol{W}}_1\\ \vdots\\ \dot{\boldsymbol{W}}_n\\ \hline (\dot{x}_p-\dot{z})\end{array}\right\}.$$

The electric enthalpy density for each appendage has the form $\mathscr{H}_i := \frac{1}{2}C_iS_i^2 - e_iS_iE_i - \frac{1}{2}\varepsilon_iE_i^2$ where $E_i = E_i(x_i)$ is the electric field in the z_i direction and $S_i = -z_i\frac{\partial^2 w_i}{\partial x_i^2}$ is the axial strain given by the kinematics of Bernoulli-Euler beam theory. The linear piezoelectric constitutive law in the i^{th} appendage of the SOA is written as

$$\begin{Bmatrix}T_i\\ D_i\end{Bmatrix} = \begin{bmatrix}C_i & -e_i\\ e_i & \varepsilon_i\end{bmatrix}\begin{Bmatrix}S_i\\ E_i\end{Bmatrix}$$

where $C_i = C_{11,i}^E$ is the stiffness coefficient, $e_i = e_{31,i}$ is the piezoelectric coefficient, $S_i = S_{11,i}$ is the axial strain in the x_i direction, $T_i = T_{11,i}$ is the axial stress in the x_i direction, $E_i = E_{3,i}$ is the electric field in the z_i direction, and $D_i = D_{3,i}$ is the electric displacement in the z_i direction. We can now write the electric enthalpy density in the i^{th} appendage as $\mathscr{V}_{\mathscr{H}i} = \int_0^{L_i} A_i\mathscr{H}_i dx_i$. Since the divergence of the electric field is zero, it is possible to express electric field in the form $\boldsymbol{E}_i = -\nabla\phi_i$ or $E_i := E_{3,i} = -\frac{\partial\phi_i}{\partial z_i}$. Assuming a linear variation in the potential ϕ_i across the thickness of the piezoelectric patch in appendage i, we obtain

$$E_i(x_i,y_i,z_i) = \begin{cases} -\frac{V_i}{t_{p,i}} & (x_i,y_i,z_i)\in \text{top patch},\\ \frac{V_i}{t_{p,i}} & (x_i,y_i,z_i)\in \text{bottom patch},\\ 0 & \text{otherwise}.\end{cases}$$

Substituting the expression for strain and the electric field, the electric enthalpy for the i^{th} appendage has the form

$$\begin{aligned}\mathscr{V}_{\mathscr{H}i} =& \frac{1}{2}\int_0^{L_i}\left(\int\int C_iz_i^2dy_idz_i\right)\left(\frac{\partial^2 w_i}{\partial x_i^2}\right)^2 dx_i\\ &+\int_0^{L_i}\left(\int\int e_iz_iE_idy_idz_i\right)\frac{\partial^2 w_i}{\partial x_i^2}dx_i\\ &-\frac{1}{2}\int_0^{L_i}\left(\int\int \varepsilon_iE_i^2dy_idz_i\right)dx_i,\end{aligned}$$

or

$$\mathscr{V}_{\mathscr{H}\,i} = \frac{1}{2}\int_0^{L_i} C_i I_i \left(\frac{\partial^2 w_i}{\partial x_i^2}\right)^2 dx_i - \int_0^{L_i} \frac{e_i \kappa_i}{t_{p,i}} \chi_{[a_i,b_i]} \frac{\partial^2 w_i}{\partial x_i^2} dx_i V_i(t) - \frac{1}{2}\frac{\varepsilon_i 2A_i(b_i-a_i)}{t_{p,i}^2} V_i^2,$$

with $(C_iI_i)(x_i) := \int\int C_i z_i^2 dy_i dz_i$ and $\kappa_{T_i} := \int\int_{A_T} z_i dy_i dz_i$ for the top patch, $\kappa_{B_i} := \int\int_{A_B} z_i dy_i dz_i$ for the bottom patch, and $\kappa_i := \kappa_{T,i} - \kappa_{B,i}$.

We substitute the Galerkin approximations of the transverse displacement of the i^{th} appendage, $w_i(t,x_i) := \sum_{k=1}^{n_i} \psi_{i,k}(x_i) W_i(t) = \mathbf{\Psi}_i^T \boldsymbol{W}_i = \boldsymbol{W}_i^T \mathbf{\Psi}_i$, into the expression for the exlectric enthalpy. We have,

$$\mathscr{V}_{\mathscr{H}\,i} = \frac{1}{2}\boldsymbol{W}_i^T \underbrace{\int_0^{L_i} C_i I_i \mathbf{\Psi}_i'' \mathbf{\Psi}_i^{'',T} dx_i}_{\boldsymbol{K}_{ii}} \boldsymbol{W}_i - \underbrace{\int_0^{L_i} \frac{\kappa_i e_i}{t_{p,i}} \chi_{[a_i,b_i]} \mathbf{\Psi}_i^{'',T} dx_i}_{\boldsymbol{B}_i^T} \boldsymbol{W}_i V_i - \frac{1}{2} \underbrace{\frac{\varepsilon_i 2A_i(b_i-a_i)}{t_{p,i}^2}}_{D_i} V_i^2.$$

The expression for the electric enthalpy for the i^{th} appendage reduces to

$$\mathscr{V}_{\mathscr{H}\,i} = \frac{1}{2}\boldsymbol{W}_i^T \boldsymbol{K}_{ii} \boldsymbol{W}_i - \boldsymbol{B}_i^T \boldsymbol{W}_i V_i - \frac{1}{2} D_i V_i^2.$$

The total electric enthalpy of the piezoelectrically actuated SOA attached to a primary structure has the form

$$\mathscr{V}_{\mathscr{H}} := \sum_{i=1}^{N} \left(\frac{1}{2}\boldsymbol{W}_i^T \boldsymbol{K}_{ii} \boldsymbol{W}_i - \boldsymbol{B}_i^T \boldsymbol{W}_i V_i - \frac{1}{2} D_i V_i^2\right) + \frac{1}{2} k_p (x_p - z)^2 .$$

The virtual work of the external applied mechanical and electrical loads is given by

$$\delta W_{nc} = F\delta(x_p - z) + \sum_{k=1}^{N} i_k \delta\lambda_k - \sum_{k=1}^{N} 2\frac{\dot{\lambda}_k}{R_k}\delta\lambda_k + \delta W_{nc,visc},$$

where F is the force applied to the primary structure, i_k is the current applied across the k^{th} piezoelectric patch, R_k is the resistance of the resistor in the shunt circuit attached to the k^{th} appendage. In order to consider the work done by the damping in the system $\delta W_{nc,visc}$, we define the Rayleigh's dissipation function as $\mathscr{F} := \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{C} \dot{\boldsymbol{q}}$ with

$$\boldsymbol{q} = \left\{\boldsymbol{W_1}\ \boldsymbol{W_2}\ \ldots\ \boldsymbol{W_N}\ (x_p - z)\right\}^T, \qquad \boldsymbol{C} = \begin{bmatrix} \mathbb{C} & \mathbb{C}_p \\ \mathbb{C}_p^T & C_p \end{bmatrix}.$$

The virtual work done by the damping in the system can be calculated using the relation $\delta W_{nc,visc,i} = \mathscr{Q}_{visc}\delta q$ with $\mathscr{Q}_{visc} = -\frac{\partial \mathscr{F}}{\partial \dot{q}}$. Hence, the expression for the virtual work done by the damping in the system, $\delta W_{nc,visc}$, is

$$\delta W_{nc,visc} = -\delta \boldsymbol{q}^T \boldsymbol{C} \dot{\boldsymbol{q}}.$$

In order to simplify the expressions of the kinetic energy, electric enthalpy and virtual work done, we define new vectors that stack the the unknown displacements, currents, voltages and flux linkages, and mass matrices in

$$\begin{aligned} \mathbb{W} &:= \left\{\boldsymbol{W}_1\ \ldots\ \boldsymbol{W}_N\right\}^T, & \mathbb{V} &:= \left\{V_1\ \ldots\ V_N\right\}^T, \\ \mathbb{i} &:= \left\{\boldsymbol{i}_1\ \ldots\ \boldsymbol{i}_N\right\}^T, & \mathbb{\Lambda} &:= \left\{\boldsymbol{\lambda}_1\ \ldots\ \boldsymbol{\lambda}_N\right\}^T, \\ \mathbb{M}_p &:= \left\{\boldsymbol{M}_{1p}\ \ldots\ \boldsymbol{M}_{Np}\right\}, \end{aligned}$$

and define associated block matrices as

$$\begin{aligned} \mathbb{M} &:= \mathrm{diag}(\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_N), \\ \mathbb{B} &:= \mathrm{diag}(\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_N), \\ \mathbb{K} &:= \mathrm{diag}(\boldsymbol{K}_{11}, \boldsymbol{K}_{22}, \ldots, \boldsymbol{K}_{NN}), \\ \mathbb{D} &:= \mathrm{diag}(D_1, D_2, \ldots, D_N), \\ \boldsymbol{\zeta} &:= diag\left(\frac{1}{R_1}, \frac{1}{R_2}, \ldots, \frac{1}{R_N}\right). \end{aligned}$$

The expressions for kinetic energy, electric enthalpy, and virtual work done by nonconservative electromechanical loads on the system will consequently be written as

$$\begin{aligned} T &= \frac{1}{2}\left(\mathbb{M}_{pp}(\dot{x}_p - \dot{z})^2 + 2(\dot{x}_p - \dot{z})\mathbb{M}_p^T \dot{\mathbb{W}} + \dot{\mathbb{W}}^T \mathbb{M} \dot{\mathbb{W}}\right), \\ \mathscr{V}_{\mathscr{H}} &= \frac{1}{2}\left(\mathbb{W}^T \mathbb{K} \mathbb{W} - 2\mathbb{V}^T \mathbb{B}^T \mathbb{W} - \mathbb{V}^T \mathbb{D} \mathbb{V} + k_p (x_p - z)^2\right), \\ \delta W_{nc} &= F\delta(x_p - z) + \mathbb{i}^T \delta\mathbb{\Lambda} - 2\dot{\mathbb{\Lambda}}^T \boldsymbol{\zeta} \delta\mathbb{\Lambda} - \delta \boldsymbol{q}^T \boldsymbol{C} \dot{\boldsymbol{q}}, \\ &= F\delta(x_p - z) + \mathbb{i}^T \delta\mathbb{\Lambda} - 2\dot{\mathbb{\Lambda}}^T \boldsymbol{\zeta} \delta\mathbb{\Lambda} - \delta(x_p - z) C_p (\dot{x}_p - \dot{z}) \\ &\quad - \delta\mathbb{W}^T \mathbb{C} \dot{\mathbb{W}} - \delta(x_p - z)\mathbb{C}^T \dot{\mathbb{W}} - \delta\dot{\mathbb{W}}^T \mathbb{C}(x_p - z). \end{aligned}$$

Finally, we can apply Hamilton's principle for piezoelectric continua coupled to ideal electrical networks. We must have

$$\delta\int_{t_0}^{t_1}\left(T - \mathscr{V}_{\mathscr{V}_{\mathscr{H}}}\right)dt + \int_{t_0}^{t_1} \delta W_{\mathscr{H},nc} dt = 0$$

for all admissible variations of the actual trajectory in electromechanical configuration space. The variational statement becomes

$$\begin{aligned}
&\int_{t_0}^{t_1}\Big\{\mathbb{M}_{pp}(\dot{x}_p-\dot{z})\,\delta(\dot{x}_p-\dot{z})+\mathbb{M}_p^T\dot{\mathbb{W}}\delta(\dot{x}_p-\dot{z})\\
&\qquad+(\dot{x}_p-\dot{z})\,\mathbb{M}_p^T\delta\dot{\mathbb{W}}+\delta\dot{\mathbb{W}}^T\mathbb{M}\dot{\mathbb{W}}\Big\}dt\\
&-\int_{t_0}^{t_1}\Big\{\delta\mathbb{W}^T\mathbb{K}\mathbb{W}-\delta\mathbb{V}^T\mathbb{B}^T\mathbb{W}-\delta\mathbb{W}^T\mathbb{B}\mathbb{V}-\delta\mathbb{V}^T\mathbb{D}\mathbb{V}\\
&\qquad-\delta\dot{\Lambda}^T\mathscr{C}\dot{\Lambda}+k_p(x_p-z)\,\delta(x_p-z)\Big\}dt\\
&+\int_{t_0}^{t_1}\Big\{F\delta(x_p-z)-2\dot{\Lambda}^T\boldsymbol{\zeta}\delta\Lambda-\delta\mathbb{W}^T\mathbb{C}\dot{\mathbb{W}}-\delta(x_p-z)\,\mathbb{C}^T\dot{\mathbb{W}}\\
&\qquad-\delta\dot{\mathbb{W}}^T\mathbb{C}(x_p-z)-\delta(x_p-z)\,C_p(\dot{x}_p-\dot{z})+\mathbb{i}^T\delta\Lambda\Big\}dt=0
\end{aligned}$$

for all admissible variations. After performing standard steps from variational calculus, we find that

$$\begin{aligned}
&\int_{t_0}^{t_1}\Big\{\delta(x_p-z)\Big(-\mathbb{M}_{pp}(\ddot{x}_p-\ddot{z})-\mathbb{M}_p\ddot{\mathbb{W}}-C_p(\dot{x}_p-\dot{z})-\mathbb{C}_p^T\dot{\mathbb{W}}\\
&\qquad\qquad-k_p(x_p-z)+F\Big)\\
&+\delta\mathbb{W}^T\Big(-\mathbb{M}_p(\ddot{x}_p-\ddot{z})-\mathbb{M}\ddot{\mathbb{W}}-\mathbb{C}\dot{\mathbb{W}}-\mathbb{C}_p(\dot{x}_p-\dot{z})-\mathbb{K}\mathbb{W}+\mathbb{B}\mathbb{V}\Big)\\
&+\delta\Lambda^T\Big(-\mathbb{B}^T\dot{\mathbb{W}}-\mathbb{D}\dot{\mathbb{V}}-2\boldsymbol{\zeta}\dot{\Lambda}+\mathbb{i}-\mathscr{C}\dot{\mathbb{V}}\Big)\Big\}dt\\
&+\text{variational BCs}=0\cdot
\end{aligned}$$

The above expression must hold for all admissible variations δx_p, $\delta\mathbb{W}$, and $\delta\lambda$ in the electromechanical configuration space. Hence, we can conclude that the strong form the solutions must satisfy the governing equations

$$\begin{bmatrix}\mathbb{M} & \mathbb{M}_p\\ \mathbb{M}_p^T & \mathbb{M}_{pp}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbb{W}}\\ \ddot{x}_p-\ddot{z}\end{Bmatrix}+\begin{bmatrix}\mathbb{C} & \mathbb{C}_p\\ \mathbb{C}_p^T & C_p\end{bmatrix}\begin{Bmatrix}\dot{\mathbb{W}}\\ \dot{x}_p-\dot{z}\end{Bmatrix}+\begin{bmatrix}\mathbb{K} & \mathbf{0}\\ \mathbf{0}^T & k_p\end{bmatrix}\begin{Bmatrix}\mathbb{W}\\ x_p-z\end{Bmatrix}-\begin{bmatrix}\mathbb{B}\\ \mathbf{0}\end{bmatrix}\mathbb{V}=\begin{bmatrix}\mathbf{0}\\ F\end{bmatrix},$$

or

$$\begin{bmatrix}\mathbb{M} & \mathbb{M}_p\\ \mathbb{M}_p^T & \mathbb{M}_{pp}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbb{W}}\\ \ddot{x}_p\end{Bmatrix}+\begin{bmatrix}\mathbb{C} & \mathbb{C}_p\\ \mathbb{C}_p^T & C_p\end{bmatrix}\begin{Bmatrix}\dot{\mathbb{W}}\\ \dot{x}_p\end{Bmatrix}+\begin{bmatrix}\mathbb{K} & \mathbf{0}\\ \mathbf{0}^T & k_p\end{bmatrix}\begin{Bmatrix}\mathbb{W}\\ x_p\end{Bmatrix}-\begin{bmatrix}\mathbb{B}\\ \mathbf{0}\end{bmatrix}\mathbb{V}=\begin{bmatrix}\mathbb{M}_p\ddot{z}+\mathbb{C}_p\dot{z}\\ F+\mathbb{M}_{pp}\ddot{z}+C_p\dot{z}+k_pz\end{bmatrix},$$

as well as the equality

$$\mathbb{B}^T\dot{\mathbb{W}}+\mathbb{D}\dot{\mathbb{V}}+2\boldsymbol{\zeta}\dot{\Lambda}-\mathbb{i}+\mathscr{C}\dot{\mathbb{V}}=\mathbf{0}\cdot$$

Figure 6 depicts the magnitude of the frequency response function from the external force input f to the primary displacement x_p when $C=6.9e10$ Pa, $\rho_m=2.7e3$ kg/m^3, $L=0.5$ m, $w=0.025$ m, $t=0.003$ m, $a=0.25$ L, $b=0.75$ L, $t_p=0.0005$ m, $e_{31}=-10.4$ C/ m^2, $\varepsilon=13.3$ nF/m, $m_p=5000$ kg, $K_{pp}=6.3665e+06$ N/m, $\boldsymbol{\zeta}=0$ ohm^{-1}, $\mathscr{C}=0$, $\mathbb{i}=0$ A. The tip masses of the appendages of SOA were chosen, from $m_i=0.03424$ kg to 0.07313 kg, such that the first resonant frequency of all the appendages are evenly distributed around the resonant frequency of the host structure.

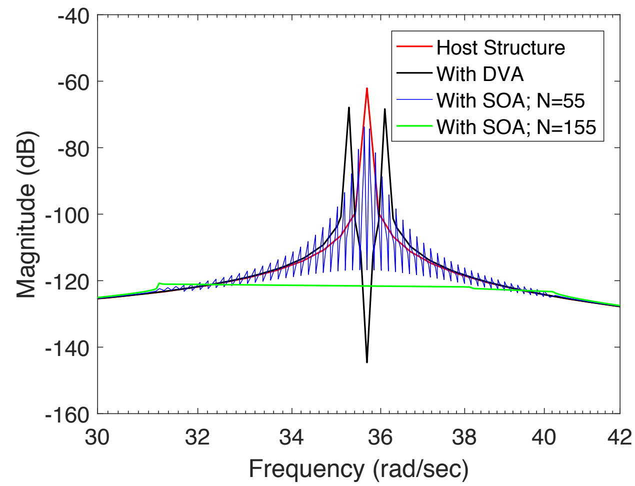

FIGURE 6: Frequency Response Function from External Force Input to Displacement of the Primary Mass

ENERGY EXTRACTION VIA SWITCHING CONTROL

In principle the piezoelectric composite SOA can be used both for passive and active vibration attenuation. In this section we discuss a strategy for energy extraction using the SOA based on a nonlinear switching controller. To illustrate the essentials of the approach, we consider only a single, base-driven SOA appendage in this brief paper. Consider the system depicted in Figure 5, but modified so that the base of the appendage at the left is driven vertically by an input motion $z(t)$ in the x_3 direction, and with the addition of a single resistor R in parallel with the capacitor C. The application of the thermodynamic variational principles in this case requires that a solution (w_3, V) satisfies the pair of equations

$$\rho_m A\left(\frac{\partial^2 w}{\partial t^2}+\ddot{z}\right)+\frac{\partial^2}{\partial x_1^2}\left(C_{11}^E I\frac{\partial^2 w}{\partial x_1^2}\right), \\ -\frac{\partial^2}{\partial x_1^2}\left(\frac{Ke_{31}\chi_{[a,b]}}{t_p}V\right)=0$$

$$\int_0^L \frac{\partial}{\partial t}\left(\frac{Ke_{31}\chi_{[a,b]}}{t_p}\left(\frac{\partial^2 w}{\partial x_1^2}\right)\right)dx_1 \\ +\left(\frac{A_p\varepsilon_{33}^S(b-a)}{t_p^2}+C\right)\dot{V}+2\frac{V}{R}=0,$$

for all $(t,x_1)\in[0,\infty)\times[0,L]$, subject to appropriate variational boundary conditions and initial conditions. With the introduction of the Galerkin approximation $w(t,x_1)=\sum_j N_j(x_1)w_j(t)$, we obtain equations of the form

$$M_{ij}\ddot{w}_j+C_{ij}\dot{w}_j+K_{ij}w_j=B_iV-P_i\ddot{z} \tag{5}$$

$$B_i^T\dot{w}_i(t)+(C_p+C)\dot{V}+2\frac{V}{R}=0 \tag{6}$$

with the mass matrix $M_{ij}:=\int_0^L\rho_m AN_j(x_1)N_i(x_1)dx_1$, the stiffness matrix $K_{ij}:=\int_0^L C_{11}^E IN_j''(x_1)N_i''(x_1)dx_1$, the control input vector $B_i:=\int_0^L\frac{Ke_{31}\chi_{[a,b]}}{t_p}N_i''(x_1)dx_1$, the generalized force $P_i:=\int_0^L\rho_m AN_i(x_1)dx_1$, and the effective capacitance of the appendage $C_p:=\frac{A_p\varepsilon_{33}^S(b-a)}{t_p^2}$.

Let us focus on Equation 6. This equation can be interpreted as the sum of currents in the circuit, where $i_p:=B_i\dot{w}_i+C_p\dot{V}$ is the current flowing through piezoelectric, and $i_c:=2C\dot{V}+\frac{2}{R}V$ is the current flowing through the shunt circuit. As the piezoelectric beam vibrates, the direction of current, i_p changes polarity. In order for the capacitor to be charged, the direction of current in the shunt circuit should remain the same. Hence a switching circuit, which has two states, is added. If the voltage across the circuit is denoted by $\tilde{V}$, we have $\tilde{V}=V$ in state 0, while in state 1, it is $\tilde{V}=-V$. The governing equations for both the switching states can be expressed as

$$\ddot{w}_j=-M_{ij}^{-1}K_{ij}w_j-M_{ij}^{-1}C_{ij}\dot{w}_j\pm M_{ij}^{-1}B_i\tilde{V}-M_{ij}^{-1}P_i\ddot{z}, \tag{7}$$

$$\dot{\tilde{V}}=-\frac{2}{R(C_p+C)}\tilde{V}\mp\frac{B_i^T\dot{w}_i}{(C_p+C)},$$

or

$$\dot{\tilde{V}}=\mp\frac{2}{R(C_p+C)}V\mp\frac{B_i^T\dot{w}_i}{(C_p+C)}. \tag{8}$$

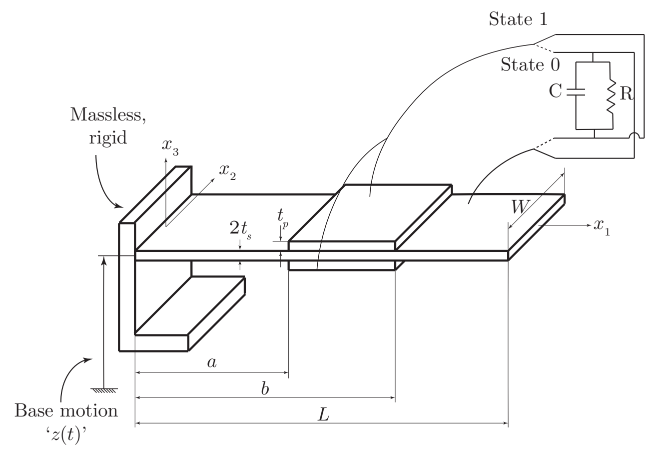

FIGURE 7: The Switch Design and Associated Circuits

A switch is introduced as depicted in Figure 7. We choose a nonlinear switching strategy that is defined in terms of the switching variable s so that the right hand side in Equation 8 remains positive during both the switching stages. We choose $s=0$ when $B_i^T\dot{w}_i+2\frac{V}{R}<0$ and $s=1$ otherwise. With this nonlinear switching strategy, the governing equations become

$$\ddot{w}_j=-M_{ij}^{-1}K_{ij}w_j-M_{ij}^{-1}C_{ij}\dot{w}_j+(-1)^sM_{ij}^{-1}B_i\tilde{V}-M_{ij}^{-1}P_i\ddot{z}, \tag{9}$$

$$\dot{\tilde{V}}=-\frac{2}{R(C_p+C)}\tilde{V}-\frac{(-1)^sB_i^T\dot{w}_i}{(C_p+C)}. \tag{10}$$

Equations 9 and 10 were simulated in Matlab when $C=6.9e10$ Pa, $\rho_m=2.7e3$ kg/m^3, $L=0.1$ m, $w=0.01$ m, $t=0.002$ m $a=0$, $b=L$, $t_p=0.001$ m, $Cp=10e-6$ F, $z=0.01sin(100t)$ m, $e_{31}=-10.4$ C/ m^2 and $R=10$ kohm. Figure 8 represents the variation of voltage across the capacitor when the switching strategy is not implemented. It is important to note that the voltage is of the order of 10^{-3} V and has the same period as the forcing frequency. Figure 9 depicts the time variation of voltage across the shunt circuit's capacitor. It is evident form the figure that the voltage across the capacitor varies as expected from an RC circuit when the switching strategy is implemented. The effectiveness of passive linear modeling and nonlinear switching to channel energy into the SOA will be discussed at the conference.

REFERENCES

[1] A. Akay, Z. Xu, A. Carcaterra, and I. Murat Koc, *Experiments on Vibration Absorption Using Energy Sinks*, *J.*

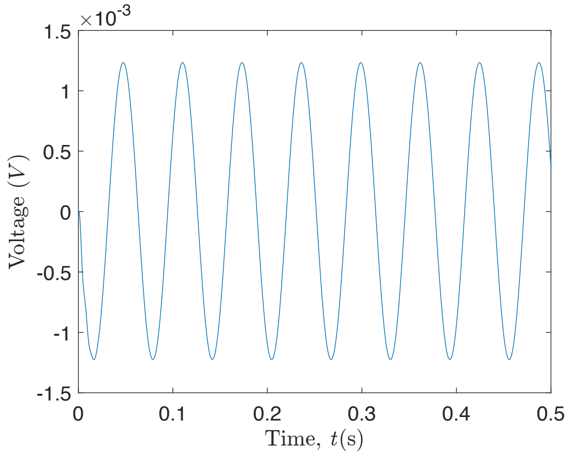

FIGURE 8: Voltage Across the Shunt Capacitor without Switching when $C = 0.0025$ Farads.

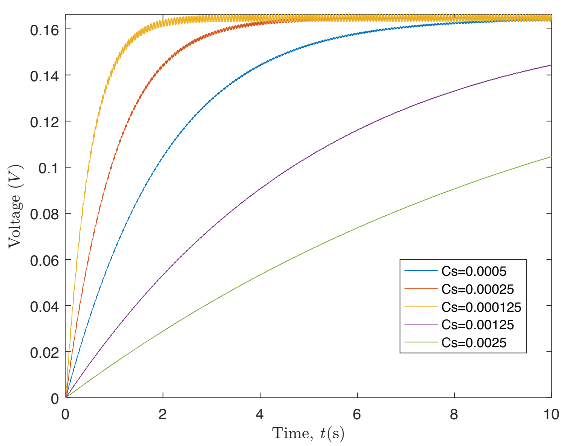

FIGURE 9: Voltage Across the Shunt Capacitor with Switching for varying Capacitances in Farads.

Acoust. Soc. Am., Volume 118, Number 5, pp. 3043-3049, 2005.

[2] A. Carcaterra, A. Akay, and C. Bernardini, *Trapping of Vibration Energy into a Set of Resonators: Theory and Applications to Aerospace Structures*, *Mechanical Systems and Signal Processing*, Volume 26, pp. 1-14, January, 2012.

[3] Christopher L. Davis, George A. Lesiutre, and Jeffrey J. Dosch, *Tunable Electrically Shunted Piezoceramic Vibration Absorber*, in *Smart Structures and Materials 1997*, pp. 51-59, *International Society for Optics and Photonics*, 1997.

[4] Woon Kyung Kim, *Design and Analysis of Switching Circuits for Energy Harvesting in Piezostructures*, PhD Dissertation, Virginia Tech, 2012.

[5] Dean C. Karnopp, Donald L. Margolis, and Ronald C. Rosenberg, *System Dynamics: Modeling, Simulation, and Control of Mechatronic Systems*, John Wiley & Sons, Inc., 2012.

[6] L. Kitis, B.P. Wang, and W.D. Pilkey, *Vibration Reduction Over a Frequency Range*, *Journal of Sound and Vibration*, Vol. 84, No. 4, pp. 559-569, August, 1983.

[7] Andrew J. Kurdila and Pablo Tarazaga, *Vibration of Linear Piezostructures*, John Wiley & Sons, Inc., in press, 2017.

[8] Donald J. Leo, *Engineering Analysis of Smart Material Systems*, John Wiley & Sons, Inc., 2007.

[9] G. Maidanik, *Induced Damping by a Nearly Continuous Distribution of Nearly Undamped Oscillators: Linear Analysis*, *Journal of Sound and Vibration*, Volume 240, pp. 717-731, 2001.

[10] Leonard Meirovitch, *Methods of Analytical Dynamics*, Macmillan, 1970.

[11] A. Preumont, *Mechatronics: Dynamics of Electromechanical and Piezoelectric Systems*, Springer, 2006.

[12] M. Strasberg and D. Feit, *Vibration Damping of Large Structures Induced by Attached Small Resonant Structures*, *J. Acoust. Soc. Am.*, Volume 99, Number 1, pp. 335-344, 1996.

[13] Harry F. Tiersten, *Linear Piezoelectric Plate Vibrations*, Plenum Press, 1969.

[14] Joseph Vignola, John Judge, and Andrew Kurdila, *Shaping of a System's Frequency Response Using an Array of Subordinate Oscillators*, *Journal of the Acoustical Society of America*, Vol. 126, pp. 129-139, 2009.

[15] Joseph Vignola, Aldo Glean, John Judge, and Teresa Ryan, *Optimal Apparent Damping as a Function of the Bandwidth of an Array of Vibration Absorbers*, *The Journal of the Acoustical Society of America*, Volume 134, Number 2, pp. 1067-1070, August, 2013.

[16] Joseph Vignola, John Judge, John Sterling, Teresa Ryan, Andrew Kurdila, Sai Tej Paruchuri, and Aldo Glean, *On the Use of Shunted Piezo-Actuators for Mitigation of Distribution Errors in Resonator Arrays*, *Proceedings of the* 22^{nd} *International Congress on Acoustics*, Buenos Aires, September 5-9, 2016.

[17] Jiashi Yang, *An Introduction to the Theory of Piezoelectricity*, Springer, 2005.

[18] Yi-Yuan Yu, *Vibrations of Elastic Plates: Linear and Nonlinear Dynamical Modeling of Sandwiches, Laminated Composites, and Piezoelectric Layers*, Springer, 1996.